Lecture 2

A brief overview of linearized gravitational waves and their interaction with nonrelativistic particles.

The progression from Newton's law of gravitation to the geodesic equation

- Newton's theory is encapsulated in the trajectory of neutral test particles

$$\frac{d^2x^i}{dt^2} + \frac{\partial \phi}{\partial x^i} = 0 \qquad (25)$$

where $x^i (i = 1, 2, 3)$ are the spatial coordinates and the source equation for the Newtonian potential ϕ is given by

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- Trajectories as geodesics A curved trajectory in flat three dimensional space. Cartan generalized this viewpoint by interpreting the trajectories as geodesics in four dimensional curved spacetime,

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{dt} \frac{dx^\rho}{dt} = 0 \tag{27}$$

This is possible if one takes $x^\mu = (x^0 = t, x^i)$ and chooses the ansatz

$$\Gamma^i_{00} = \frac{\partial \phi}{\partial x^i} \quad , \quad \text{all other } \Gamma^\mu_{\nu\rho} \text{ vanish} \tag{28}$$

Here are some geometric elements related to curvature:

- Reimann Curvature Tensor

$$R^{\alpha}{}_{\beta\gamma\delta} = \partial_{\gamma}\Gamma^{\alpha}_{\beta\delta} - \partial_{\delta}\Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta} - \Gamma^{\alpha}_{\mu\delta}\Gamma^{\mu}_{\beta\gamma} \tag{29}$$

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- Ricci tensor

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} \tag{30}$$

- In particular

$$\Gamma^{i}_{00} \to R^{i}_{0j0} = \partial_i \partial_j \phi \implies R^{i}_{0i0} = R_{00} = \vec{\nabla}^2 \phi \tag{31}$$

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- Geometric Interpretation of Newtonian Gravity From eq(26), we can write

$$R_{00} = \frac{4\pi G}{c^2} T_{00} \tag{33}$$

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- **In non-relativistic (NR) spacetime, there isn't a single non-degenerate spacetime metric. However, it's important to note that although we don't require a metric tensor $g_{\mu\nu}(x)$ to calculate curvature components $R^{\alpha}{}_{\beta\gamma\delta}$, having a connection $\Gamma^{\mu}{}_{\alpha\beta}$ is sufficient for calculating curvature.**

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- **A geometric theory of non-relativistic Newtonian gravitation, with a covariant approach, emerged sometime after Einstein formulated his general theory of relativity. This was proposed by Cartan in 1923. $\implies Newton - Cartan\ Geometry$.**
- **Here, we aim to delve into the geometric formulation of relativistic spacetime.**
- **This we now wish to rewrite ($R_{00} \sim T_{00}$) in a way that is covariant under general space-time coordinate transformations.**

Gravity: Covariant Geodesics

- Treating spacetime on an equal footing (part of configuration space):

$$t = t(\tau), x^i(\tau) :=> x^\mu(\tau) = (ct(\tau), x^i(\tau))$$

with $\tau = affine\ parameter \implies t = a\tau + b$

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- Geodesic equation from the action principle

$$I^{particle}_{curved} = -m_0 c\int d\tau\sqrt{g_{\mu\nu}(x(\tau))\dot{x}^\mu(\tau)\dot{x}^\nu(\tau)} \quad (35)$$

$\delta I^{particle}_{curved} = 0 \implies eq(6)$ where

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\sigma\beta}[\partial_\alpha g_{\mu\beta} + \partial_\mu g_{\beta\alpha} - \partial_\beta g_{\mu\alpha}] \quad (36)$$

The Einstein Equivalence Principle

- The effects of any gravitational field vanish in local inertial frames.

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where q^μ is the vector connecting the corresponding point of adjacent geodesic x^μ.

Einstein Equation

- Generalization of geometric form of Newton's law of gravitation

$$R_{00} = 4\pi G T_{00},$$

were we've set $c = 1$ for brevity.

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$$R_{\mu\nu} \sim T_{\mu\nu} \tag{38}$$

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- Most general case

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- <u>At non-relativistic Limit</u>

$$R_{00} = (A + B)T_{00} - BT_{ii} \tag{41}$$

It is important to realize that in the Newtonian limit, $T_{ii} = 0$.

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$\nabla^{\mu}G_{\mu\nu} = 0 \implies (B + \frac{1}{2}A)\partial_{\nu}T^{\alpha}_{\alpha} = 0 \implies B = -\frac{A}{2}$

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$$A = 8\pi G, B = -4\pi G$$

Linearized Gravitational Waves: General formulation

- Einstein Equation

$$R_{\mu\nu} = 8\pi G \, T_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} \implies G_{\mu\nu} = 8\pi G \, T_{\mu\nu} \tag{43}$$

- How would you define the energy-momentum tensor $T^{\mu\nu}(x)$?

$$T^{\mu\nu}(x, x(\tau)) = -\frac{2}{\sqrt{-g}} \frac{\delta I^{particle}_{curved}[x(\tau)]}{\delta g_{\mu\nu}(x)} = m_0 \int d\tau \dot{x}^\mu(\tau) \dot{x}^\nu(\tau) \frac{\delta^4(x - x(\tau))}{\sqrt{-g(x)}} \tag{44}$$

with $g(x) = det(g_{\mu\nu}(x))$

Linearized/ Weak gravity

- Gravitational Action If the $T_{\mu\nu} = 0$ is zero, that essentially corresponds to a vacuum, and the equation becomes

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

- Can we derive this the the action Principle?

$$S_g = \frac{1}{16\pi G} S_{EH} \tag{45}$$

with

$$S_{EH}[g_{\mu\nu}(x)] = \int d^4x \sqrt{-g} R, \tag{46}$$

- Linearized version of GR

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x); \quad | h_{\mu\nu} | << 1 \tag{47}$$

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- The Christoffel connection and Riemann curvature in the linearized metric :

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2}\eta^{\mu\rho}(\partial_\sigma h_{\nu\rho} + \partial_\nu h_{\sigma\rho} - \partial_\rho h_{\nu\sigma} - \partial_\sigma\partial_\lambda h_{\nu\rho}) \tag{49}$$

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$$S_{EH} = \frac{1}{64\pi G}\int d^4x \, (h_{\mu\nu}\Box h^{\mu\nu} + 2h^{\mu\nu}\partial_\mu\partial_\nu h - h\Box h - 2h_{\mu\nu}\partial_\rho\partial^\mu h^{\nu\rho}) \quad (51)$$

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Here ξ_μ are completely arbitrary except that they are considered to be small.

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- The field equations for $h^{\mu\nu}$ in vacuum :

$$\Box h^{\mu\nu} + \partial^\mu\partial_\alpha h^{\alpha\nu} + \partial^\nu\partial_\alpha h^{\alpha\mu} - \partial^\mu\partial^\nu h + \eta^{\mu\nu}(\Box h - \partial_\alpha\partial_\beta h^{\alpha\beta}) = 0. \quad (54)$$

Complete Gauge (Harmonic and residual) Fixing

- Transverse-traceless (TT) gauge: The metric perturbation obeys

$$h_{0\mu} = 0, \quad \partial^j h_{ij} = 0; \; h^{\mu}{}_{\mu} = 0 \tag{55}$$

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- Plane waves solutions

$$h_{ij}(x) = Re[\epsilon_{ij}\; e^{ikx}] \tag{57}$$

$$\partial^j h_{ij} = 0 \implies k^j \epsilon_{ij} = 0 \tag{58}$$

If we consider GWs propagating z direction, then we have

$$\mathcal{E} \equiv \{\varepsilon_{ij}\} = \begin{pmatrix} \epsilon_+ & \epsilon_\times & 0 \\ \epsilon_\times & -\epsilon_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \tag{59}$$

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- Comment: Due to transversality condition, h_{ij} has non zero components in the x-y plane. And ϵ_+ and ϵ_x are called "+" ($h_{11} = -h_{22}$) and "×" polarization ($h_{12} = h_{21}$) of the GWs respectively.

NR test particles

- Geodesic deviation

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where q^μ is the vector connecting the corresponding point of adjacent geodesic x^μ.

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$$\frac{D^2 q^\mu}{D\tau^2} = -R^\mu_{\nu\rho\sigma} q^\rho \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} \tag{60}$$

where q^μ is the vector connecting the corresponding point of adjacent geodesic x^μ.

- NR test particles: slow velocity

If we consider slowly moving test particles $\implies \frac{dx^\nu}{d\tau} = U^\mu \sim (c, 0, 0, 0) + \mathcal{O}(h)$. As slowly moving particle we have $d\tau = dt$. In NR limit, spatial components of the separation four vectorq^μ reduce to :

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- Long wavelength approximation limit:
  - GWs propagating along x_3 direction $\implies h_{ij} \neq 0$ for $i,j = 1,2$.
  - Long wavelength limit: $e^{i\vec{k}.\vec{x}} \sim 1 \implies$ GWs can then treated as a function of time only.

Interaction between particles and GWs

- <u>Observation</u>
  • At TT gauge, long wavelength approximation, the whole analysis effectively is described by Newtonian mechanics. And the components of gravitational waves in TT gauge which produces a "tidal" effect in the equation of motion of the given mass.

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- Mechanical detector

  Consider GWs incident on a detector (composed of two masses in presence of interacting via a mechanical potential):

  If GWs propagate along the direction normal to the oscillating plane.

$$m\frac{d^2 q_i}{dt^2} = \frac{m}{2}\ddot{h}_{ij}(t)q_j + \partial_i V(q_i) \quad i,j = 1,2 \tag{63}$$

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- Lagrangian

$$L = \frac{1}{2} m \dot{q}_i^2 - \frac{1}{2} m \dot{h}_{jk}(t) \dot{q}_j q_k - V(q_i) \tag{64}$$

- Hamiltonian

$$H_{ho\ gw} = \frac{1}{2m}(p_j + \frac{1}{2} m \dot{h}_{jk} q_k)^2 + V(q_i) \tag{65}$$

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- **Gravitational waves can be viewed as the vacuum solution of the linearized Einstein equations in the transverse-traceless (TT) gauge, representing a distortion of flat spacetime when observed far away from the source.**
- **We've developed a mechanical model that allows us to analyze the interaction between neutral particles and gravitational waves.**

THANK YOU!