Exploring Quantum Aspects of Gravitational Waves

NITheCS MINI-SCHOOL

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Plan of the NITheCS Mini School

- **Lectute 1: Geometrization of the classical mechanics**

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- **Lecture 2: A brief overview of linearized gravitational waves and their interaction with nonrelativistic particles.**

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- **Observations and conclusions**

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- **Due to the very small length-scale at which GWs interact matter – characterized by the dimensionless strain amplitude ($h \sim \frac{\delta L}{L} \sim 10^{-21}, L \sim 1km\ , \delta L \sim 10^{-18}m$)-it becomes apparent that the manifestation of experimental evidence for gravitational waves is anticipated at the quantum mechanical level.**

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- **The (GWs+matter)interaction must be treated quantum mechanical in nature.**
- **This opens up a new avenue for detecting the quantum nature of gravity as proposed by Frank Wilczek in PRL 2021.**

Lecture 1
Geometrization of the classical mechanics

Dynamics to Geodesic Equation

- Newton's Law of motion: Dynamics

$$m(\frac{d^2x^a}{dt^2}) + \frac{\partial U}{\partial x^a} = 0,\ \ a = 1, 2, ...n \tag{1}$$

where x^a are the spatial coordinates flat Euclidean space: $\sqrt{\delta_{ab}dx^a dx^b}$, and $U(x)$ is the potential energy.

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- Lagrangian Picture: Action Functional

$$S[x^a] = \int_{t_i}^{t_f} dtL(x^a.\dot{x}^a; t) \tag{3}$$

How kinetic energy (T) and potential energy (U) change as a particle moves along its trajectory?

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- Principle of least action

$$\delta S[x^a] = \int_{t_i}^{t_f} dt\ (\delta L(x^a.\dot{x}^a; t)) = 0 \quad (4)$$

Calculation

Euler Lagrangian Equation of Motion

- Under an arbitrary variation of the action along the trajectory

$$\delta S = -\int_{t_i}^{t_f} dt \, [\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}^a}) - \frac{\partial L}{\partial \dot{x}^a}]\delta x^a = 0 \tag{5}$$

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- Euler Lagrange EOM

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}^a}) - \frac{\partial L}{\partial \dot{x}^a} = 0 = m(\frac{d^2 x^a}{dt^2}) + \frac{\partial U}{\partial x^a} \implies x^a = x^a(t)$$

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- Now, if we compare with eq (1) arrive at

$$L = \frac{1}{2}m(\delta_{ab}\dot{x}^a\dot{x}^b) - V(x^a) \tag{6}$$

Calculation

A curved trajectory interpreting as geodesics

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- For potential motion, a curved trajectory in a flat, three-dimensional space.
- Can we reinterpret the potential motion in flat space as a free motion in curved Riemann space?

Potential motion as a geodesic of a Reimann space

- Let us consider the solution to EOM: $x^1 = x^1(t) \implies t := t(x^1)$ (Invertable function)

$$\frac{dx^a}{dt} = x^{,a}(\frac{dx^1}{dt}); \quad \frac{d^2x^a}{dt^2} = x^{,,a}(\frac{dx^1}{dt})^2 + x^{,a}\ddot{x}^1$$

with $x^{,a} = \frac{dx^a}{dx^1}, x^{,,a} = \frac{d^2x^a}{dx^1dx^1} \quad a = 1, 2, 3, ...n$

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- Repramatrization of EOM: $x^i = x^i(x^1)$

$$\frac{d^2x^i}{dt^2} + \frac{\partial U}{\partial x^i} = 0, \quad i = 2, ...n \tag{7}$$

with $m = 1\ unit$

$$\implies x^{,,i}(\dot{x}^1)^2 + \frac{\partial U}{\partial x^i} - x^{,i}\frac{\partial U}{\partial x^1} = 0 \tag{8}$$

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$$\implies x^{,,i}(\dot{x}^1)^2 + \frac{\partial U}{\partial x^i} - x^{,i}\frac{\partial U}{\partial x^1} = 0 \tag{8}$$
- Energy conservation
$$(\dot{x}^1)^2\frac{1}{2}(x^{,a})^2 + U(x^a) = E \implies \frac{1}{\dot{x}} = \frac{dt}{dx^1} = \sqrt{\frac{\delta_{ab}x^{,a}x^{,b}}{2(E-U)}} \tag{9}$$

Geodesics: Geometry

- After reparametrization of the dynamical EOM:

$$x^{,,i} + \left(\frac{\delta_{ab} x^{,a} x^{,b}}{2(E-U)}\right)\left(\frac{\partial U}{\partial x^i} - x^{,i}\frac{\partial U}{\partial x^1}\right) = 0 \implies What\ is\ the\ geometrical\ meaning? \quad (10)$$

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- Line element in Riemann space with metric $g_{ab}(x)$is given by

$$ds^2 = g_{ab}(x) dx^a dx^b \implies I_{curved} = \int dt\ \frac{1}{2} g_{ab}(x) \dot{x}^a \dot{x}^b \tag{11}$$

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- Euler Lagrangian EOM: Geodesic equation

$$\delta I_{curved} = 0 \implies \ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0; \quad a, b, c = 1, 2, 3...n \tag{12}$$

with

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}[\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}] \implies Riemann\ connection$$

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- Under reparametrization

$$x^{,,i} + \hat{\Gamma}^i{}_{bc}x^{,b}x^{,c} = 0; \quad i = 2, 3, ...n \quad (14)$$

with $\hat{\Gamma}^a{}_{bc} = \Gamma^a{}_{bc} - x^{,a}\Gamma^1{}_{bc}$.

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- From potential motion

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$\Gamma^a{}_{bc} = -\frac{1}{2(E-U)}(\delta^a{}_c\partial_b U + \delta^a{}_b\delta_c U - \delta^b{}_c\partial_a U)$

Conserved Quantity in Riemann space

- From free motion in Curved Riemann space

$$\frac{d}{dt}(g_{ab}(x)\dot{x}^a\dot{x}^b) = 0 \implies g_{ab}(x)\dot{x}^a\dot{x}^b = v^2 \tag{16}$$

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$$v = \sqrt{g_{ab}(x)\dot{x}^a\dot{x}^b}$$
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$$g_{ab}(x) = \frac{v^2}{2(E-U(x))}\delta_{ab} \tag{18}$$

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- From observation we can identify:

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Duality between Potential motion in flat space and free motion in curved space

- Riemann connection from metric: For $g_{ab}(x) = \frac{v^2}{2(E-U(x))}\delta_{ab}$, the Riemann connection

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- Consistency between two form of the connection demand that $v = 1$
- Free motion in Curved space can be described completely

$$I_{curved} = \int dt\ [\frac{m}{2}g_{ab}(x)\dot{x}^a\dot{x}^b] \implies Geometric\ Action \tag{21}$$

with $g_{ab}(x) = \frac{\delta_{ab}}{2(E-U(x))}$

Gravity: Covariant Geodesics

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$$t = t(\tau), x^i(\tau) :=> x^\mu(\tau)$$

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- Geodesic equation from the action principle

$$I^{particle}_{curved} = -m_0c\int\sqrt{g_{\mu\nu}(x)dx^\mu dx^\nu} \quad (23)$$

$\delta I^{particle}_{curved} = 0 \implies eq(6)$ where

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\sigma\beta}[\partial_\alpha g_{\mu\beta} + \partial_\mu g_{\beta\alpha} - \partial_\beta g_{\mu\alpha}] \quad (24)$$

What insights can be drawn from our today's discussion?

- We demonstrate that the system's configuration space can be endowed with a metric, which is constructed using a potential.

$$lim_{U\to 0} g_{ab}(x) = lim_{U\to 0}[\frac{\delta_{ab}}{2(E-U(x))}] \to \delta_{ab}$$

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- Can the concept of geometry be extended to include velocity-dependent potential terms, such as those encountered by a charged particle in a magnetic field? Take some time to consider this idea.

THANK YOU!