Green's Functions for Ordinary Differential Equations

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MINI-SCHOOL
NITheCS
South- Africa
05, 12, 19, 26, October 2022

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NITheCS Mini-course: 05, 12, 19, 26 October 2022

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- 05 October 2022 [Session 1]: Generalities about second order ordinary differential equation.

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- 26 October 2022 [Session 4]: Examples: solving some cases.

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- Because of the linearity of the [HE], a linear combination of the fundamental set of solutions is also solution of the [HE].

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The most general solution of the Homogeneous second order linear differential equation [HE] is of the form $u(x) = c_1 u_1(x) + c_2 u_2(x)$, where c_1, c_2 are arbitrary complex constants and $u_1(x), u_2(x)$ are a fundamental set of solutions.

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Theorem 2

The most general solution of the second order non-homogeneous differential equation

$$a(x)\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u = f(x),\ a(x) \neq 0,\ x \in [a, b]. \quad (3)$$

is $u(x) = c_1 u_1(x) + c_2 u_2(x) + u_p(x)$, where u_p is any particular solution of the non-homogeneous equation, u_1, u_2 are a fundamental set of solutions of the associated homogeneous equations and $c_1,\ c_2$ are arbitrary complex constants-

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- Conclusion: A knowledge of one solution of a linear [HE] of the second-order is sufficient to find the most general solution of the inhomogeneous equation.
- The method of variation of constants cannot be easily generalized either to equations of higher order or to PDEs. For this reason, the method of Green's function is introduced.

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- The integral operator:

$$K = \int_a^b \int_a^b dx'' dx' |x''\rangle w(x'') K(x'', x') w(x') \langle x'| \tag{6}$$

The integral operator is completely continuous and Hermitian if

$$K(x, x') = \bar{K}(x, x'); \quad \int_a^b \int_a^b |K(x, x') w(x) w(x') dx dx' < \infty \tag{7}$$

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- Differential operators are, in general, not bounded. However, a differential operator L may have an inverse operator L^{-1}, which is not only bounded but completely continuous.
- L^{-1} is then an integral operator.
- The most important differential operators which occur in physical problems are of second order.