Green's Functions for Ordinary Differential Equations

Dr. Laure Gouba

Abdus Salam International Centre for Theoretical Physics, Trieste, Italy
Email: laure.gouba@gmail.com

MINI-SCHOOL
NITheCS
South- Africa
05, 12, 19, 26, October 2022

NITheCS Mini-course: 05, 12, 19, 26 October 2022

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- 26 October 2022 [Session 4]: Examples: solving some cases.

Summary of Session 3 [19 October 2022]

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Problem

Find the solution of the non-homogeneous linear differential equation

$$L_x u(x) = f(x), \quad x \in [a, b], \tag{1}$$

with homogeneous boundary conditions imposed on $u(x)$ of the type:

$$B_1[u] = \alpha_1 u(a) + \beta_1 \frac{du}{dx}|_{x=a} + \gamma_1 u(b) + \delta_1 \frac{du}{dx}|_{x=b} = 0; \tag{2}$$

$$B_2[u] = \alpha_2 u(a) + \beta_2 \frac{du}{dx}|_{x=a} + \gamma_2 u(b) + \delta_2 \frac{du}{dx}|_{x=b} = 0. \tag{3}$$

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We are interested in solving this equation by the method of Green's function.

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- Step 2: Since except at the point $x = y$, the Green's function $G(x, y)$ satisfies the homogeneous equation $L_x G(x, y) = 0$ in the entire interval [a,b], we can express $G(x, y)$, in the intervals to the left and to the right of the point $x = y$ as

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$$G(x,y) = \begin{cases} c_1(y)u_1(x) + c_2(y)u_2(x) & \text{for } a \leq x < y \\ d_1(y)u_1(x) + d_2(y)u_2(x) & \text{for } y < x \leq b \end{cases} \quad (4)$$

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- Step 3: Determine the coefficients $c_1(y),\ c_2(y),\ d_1(y),\ d_2(y)$, that are arbitrary constants with respect to the variable x and may depend on the variable y.

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- G(x,y) must be continuous at x= y.
- The derivative $\frac{\partial}{\partial x}G(x, y)$ is discontinuous at the point $x = y$, hence at that point should have a jump of magnitude $\frac{1}{a(y)w(y)}$.

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NB: Demonstrations can be found in the reference book.

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Theorem

Consider the second order linear equation

$$L_x u = f(x) \tag{7}$$

with homogeneous boundary condition

$$B_1(u) = 0; \quad B_2(u) = 0. \tag{8}$$

Provided the homogeneous equation $L_x u = 0$ has no nontrivial solutions satisfying the boundary conditions (8), the Green's function associated with the equation (7) exists and is unique. The solution of the equation (7) given by

$$u(x) = \int_a^b dy w(y) G(x, y) f(y) \tag{9}$$

is unique.

Missing items

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- We consider a formal differential operator L_x and sufficiently differentiable functions u(x) and v(x), $x \in [a, b]$. Suppose that there exists a formal differential operator L_x^+ with the property that the following relation holds

$$w[\bar{v} L_x u - u\overline{L_x^+ v}] = \frac{d}{dx}\{Q[u, \bar{v}]\}, \qquad (10)$$

where w(x) is some positive definite function over $x \in [a, b]$, $Q[u, \bar{v}]$ is a function which is bilinear on u, $\bar{v}$ and their derivatives $\frac{du}{dx}$, $\frac{d\bar{v}}{dx}$.

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- In general, $Q[u, \bar{v}] = A(x)u\bar{v} + B(x)u\frac{d\bar{v}}{dx} + C(x)\frac{du}{dx}\bar{v} + D(x)\frac{du}{dx}\frac{d\bar{v}}{dx}$.

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- When $L_x = L_x^+$, the formal differential operator L_x is said to be self-adjoint.

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- Integrating the Lagrange identity (10) over the interval $[a, b]$, we get

$$\int_a^b dx\ w[\bar{v} L_x u] - \int_a^b dx\ w[u \overline{L_x^+ v}] = Q[u, \bar{v}]|_{x=b} - Q[u, \bar{v}]|_{x=a}. \quad (11)$$

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- Exercises
  1. Given the differential operator $L_x = \frac{d}{dx}$, use the generalized Green's identity to determine the adjoint operator L_x^+ with respect to a weight $w = 1$.
     Answer: $L_x^+ = -\frac{d}{dx}$.
  2. Given the differential operator $L_x = i\frac{d}{dx}$, determine the adjoint operator L_x^+ with respect to a weight $w = 1$.
     Answer: $L_x^+ = i\frac{d}{dx} = L_x$ (here L_x is self-adjoint.)

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$$L_x u(x) = f(x), \quad x \in [a, b], \tag{12}$$

where

$$L_x = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x), \; a(x) \neq 0.$$

and

$$B_1(u) = \sigma_1; \quad B_2(u) = \sigma_2.$$

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$$B_1(u) = \sigma_1; \quad B_2(u) = \sigma_2.$$

We know how to use the method of the Green's function in the case of homogeneous boundary conditions. How to proceed in situation of non homogeneous boundary conditions or in the case of initial value problem?

Missing items

1. Procedure 1: Look for a particular solution $u_p(x)$ of the homogeneous equation (HE) satisfying the non-homogeneous boundary conditions $B_1(u) = \sigma_1, \ B_2(u) = \sigma_2$.
   $u_p(x) = \alpha u_1(x) + \beta u_2(x)$, where $u_1(x), u_2(x)$, are the fundamental set of solutions of the homogeneous equation.

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The solution of (12) is then

$$u(x) = \int_a^b dy G(x,y) w(y) f(y) + u_p(x), \tag{13}$$

where $G(x,y)$ is the Green's function associated to

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2. Procedure 2: Use the generalized Green's identity to derive the surface term due to the boundary condition.