NITHeP Mini-school on quantum computing

INTRODUCTION TO THE THEORY OF QUANTUM COMPUTING

Daniel K. Park | dkp.quantum@gmail.com

KAIST

Outline

Part I: What & Why

- Introduction & Background

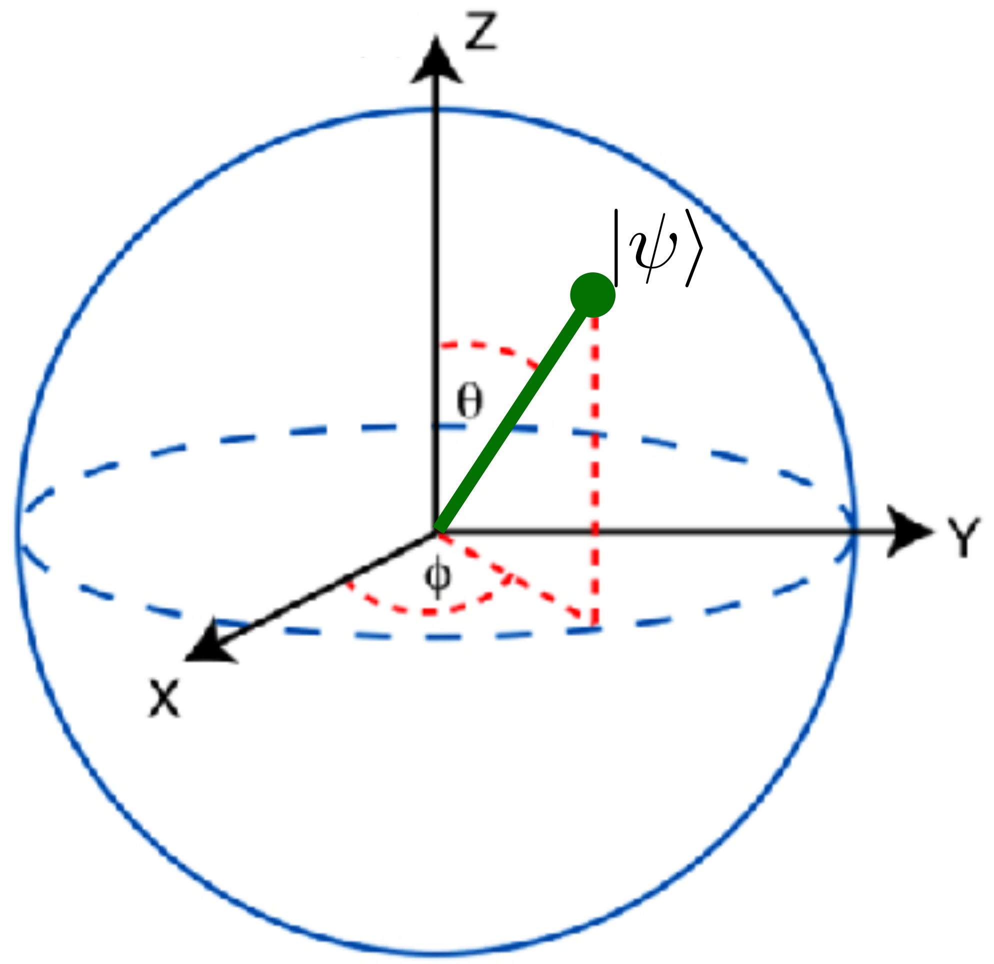
Part II: How

- Quantum Circuit
- Quantum Algorithms
- Quantum Error Correction

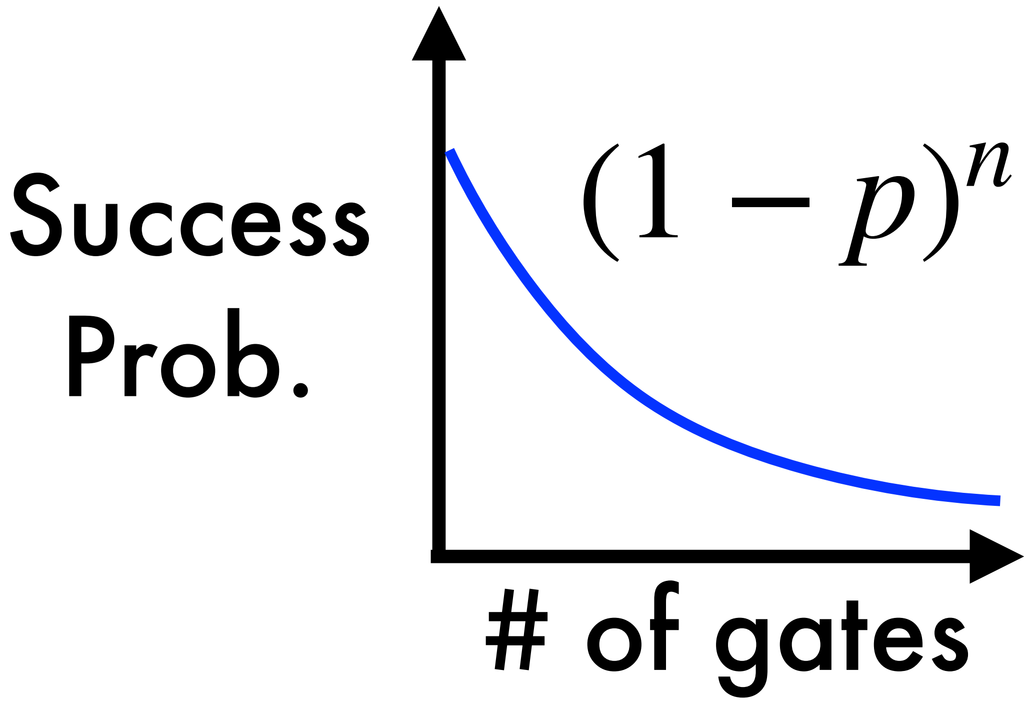
Quantum Computing Is Hard in Practice

- Classical digital computation: Very robust to noise (bit-flip error).
- Quantum: Protect not only against bit-flip errors, but also from the environment constantly interacting to the quantum system.
- But qubits need to interact strongly and accurately with each other.
- At first glance, quantum computers resembles classical analog computers

$$|\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\phi}\sin(\theta/2)\,|1\rangle.$$



- Let $\Pr(gate) = 1 - p$, then



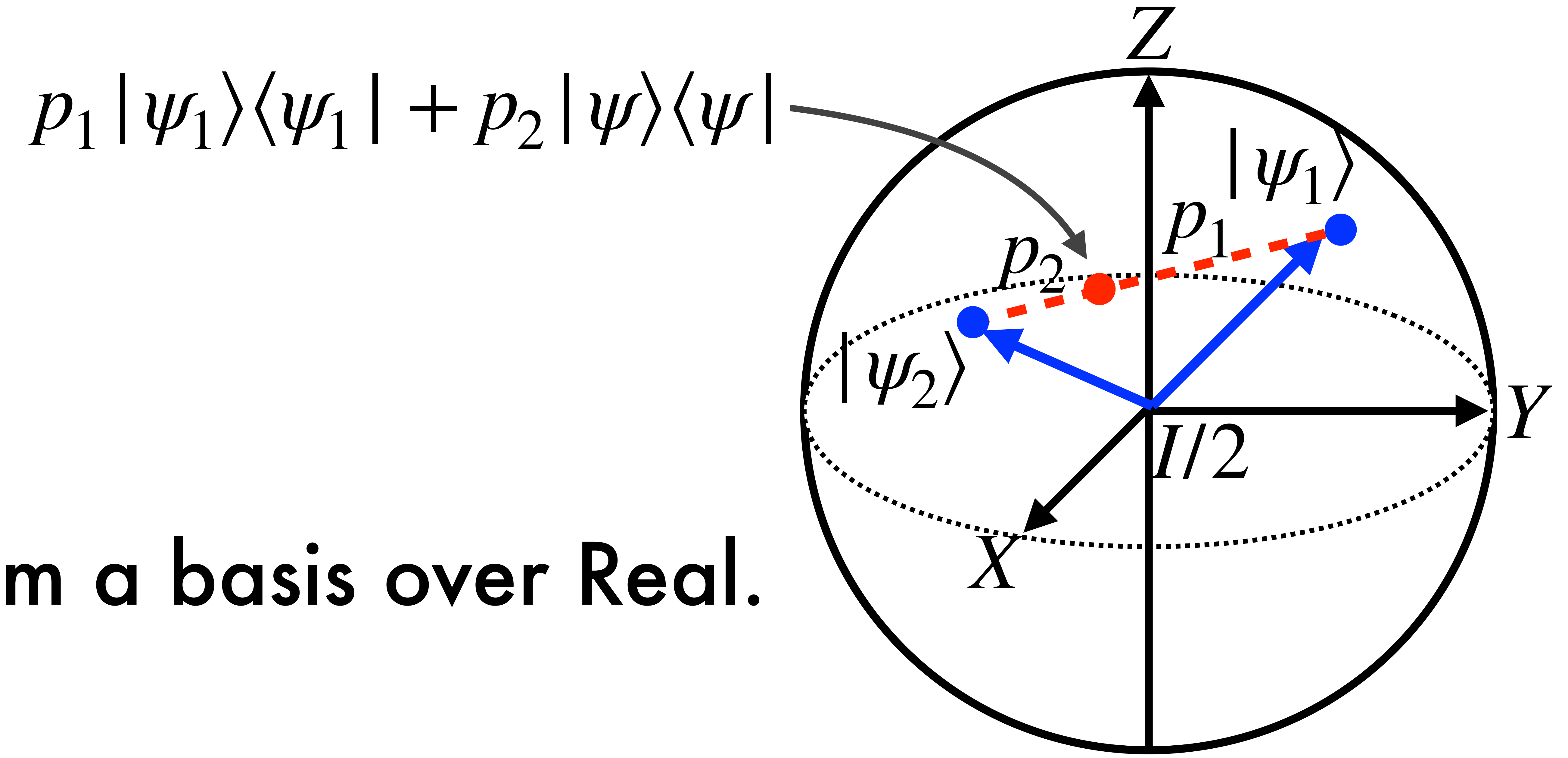
Density Matrix Formalism

- Quantum mechanics can also be formulated with density operator or density matrix, which is mathematically equivalent to the state vector approach.
- Useful for describing quantum systems whose state is not completely known
- For an ensemble of pure states $\{p_i, |\psi_i\rangle\}$, the density matrix is defined by

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

- If $\rho = |\psi\rangle\langle\psi|$, it is a pure state. Otherwise, it is a mixed state.
- $\langle\phi|\rho|\phi\rangle \geq 0$, $Tr(\rho) = 1$ and $Tr(\rho^2) \leq 1$ (equal iff ρ is a pure state).

Revisit Bloch Sphere



- Four Pauli matrices $\{I, X, Y, Z\}$ form a basis over Real.
- $\rho = \frac{1}{2}\left(I + a_x X + a_y Y + a_z Z\right),\ a_x, a_y, a_z \in \mathbb{R},\ a_x^2 + a_y^2 + a_z^2 \leq 1.$
- $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$

  ➡ $\rho = \frac{1}{2}\left(I + \sin(\theta)\cos(\phi)X + \sin(\theta)\sin(\phi)Y + \cos(\theta)Z\right)$

General Quantum Operation

A quantum operation: map that sends a density matrix to a density matrix.

- Linear: $\Lambda(p\rho + q\sigma) = p\Lambda(\rho) + q\Lambda(\sigma)$
- Preserve trace: $Tr\left(\Lambda(\rho)\right) = Tr\left(\rho\right)$
- Preserve positivity: $\rho \geq 0,\ \Lambda(\rho) \geq 0$
- Completely positive: $(\Lambda \otimes I)\,\rho \geq 0$

General Quantum Operation

The following are equivalent:

- Λ is completely-positive, trace-preserving (CPTP).
- $\exists\ U$ such that $\Lambda\left(\rho\right) = Tr_E\left(U\rho \otimes \sigma_E U^\dagger\right)$ *Motivates the use of unitary circuits
- $\Lambda(\rho) = \sum_i A_i \rho A_i^\dagger,\ \sum_i A_i^\dagger A_i = I$. A_i is called a Kraus operator.

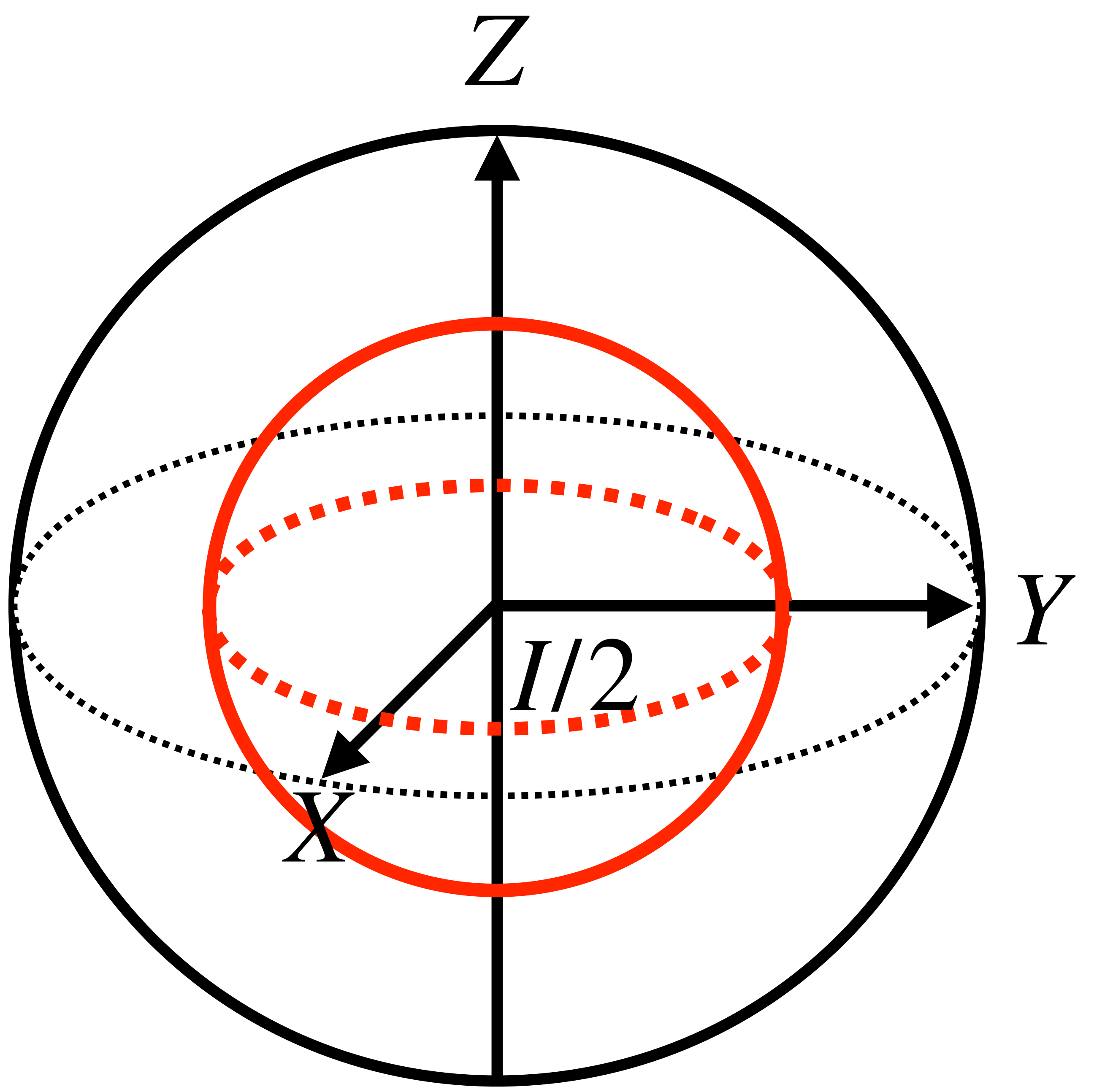
Depolarizing Channel

- $\Lambda(\rho) = (1-p)\rho + \frac{p}{3}\left(X\rho X + Y\rho Y + Z\rho Z\right).$

- Kraus operators: $\sqrt{1-p}I, \sqrt{\frac{p}{3}}X, \sqrt{\frac{p}{3}}Y, \sqrt{\frac{p}{3}}Z.$

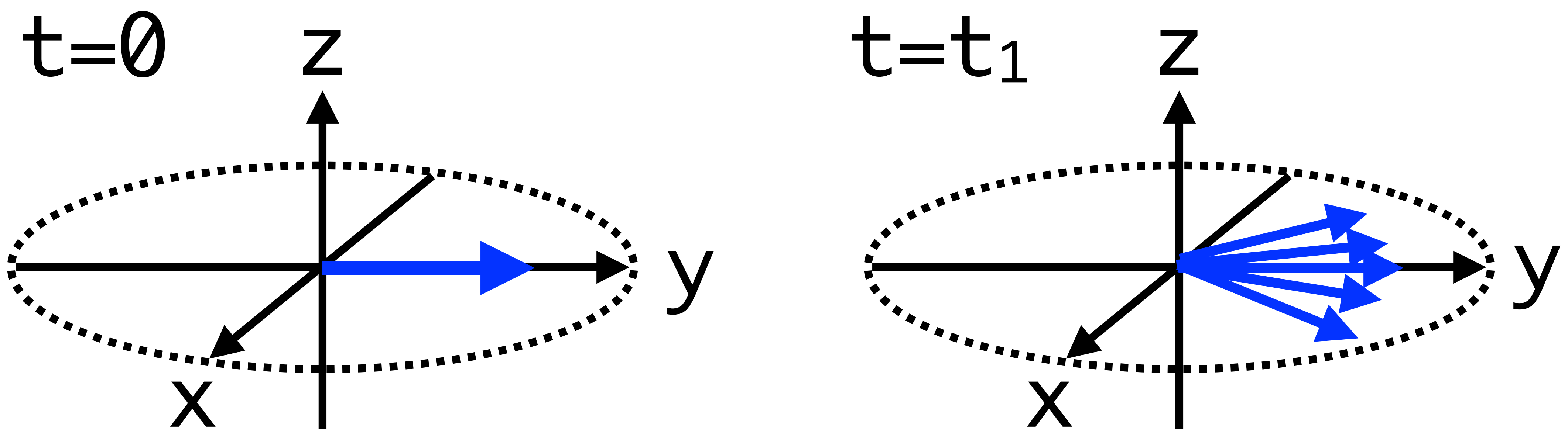
- Can also be written as $\Lambda(\rho) = (1-p')\rho + p'\frac{I}{2}.$

- Any single qubit error can be transformed to depolarizing error.



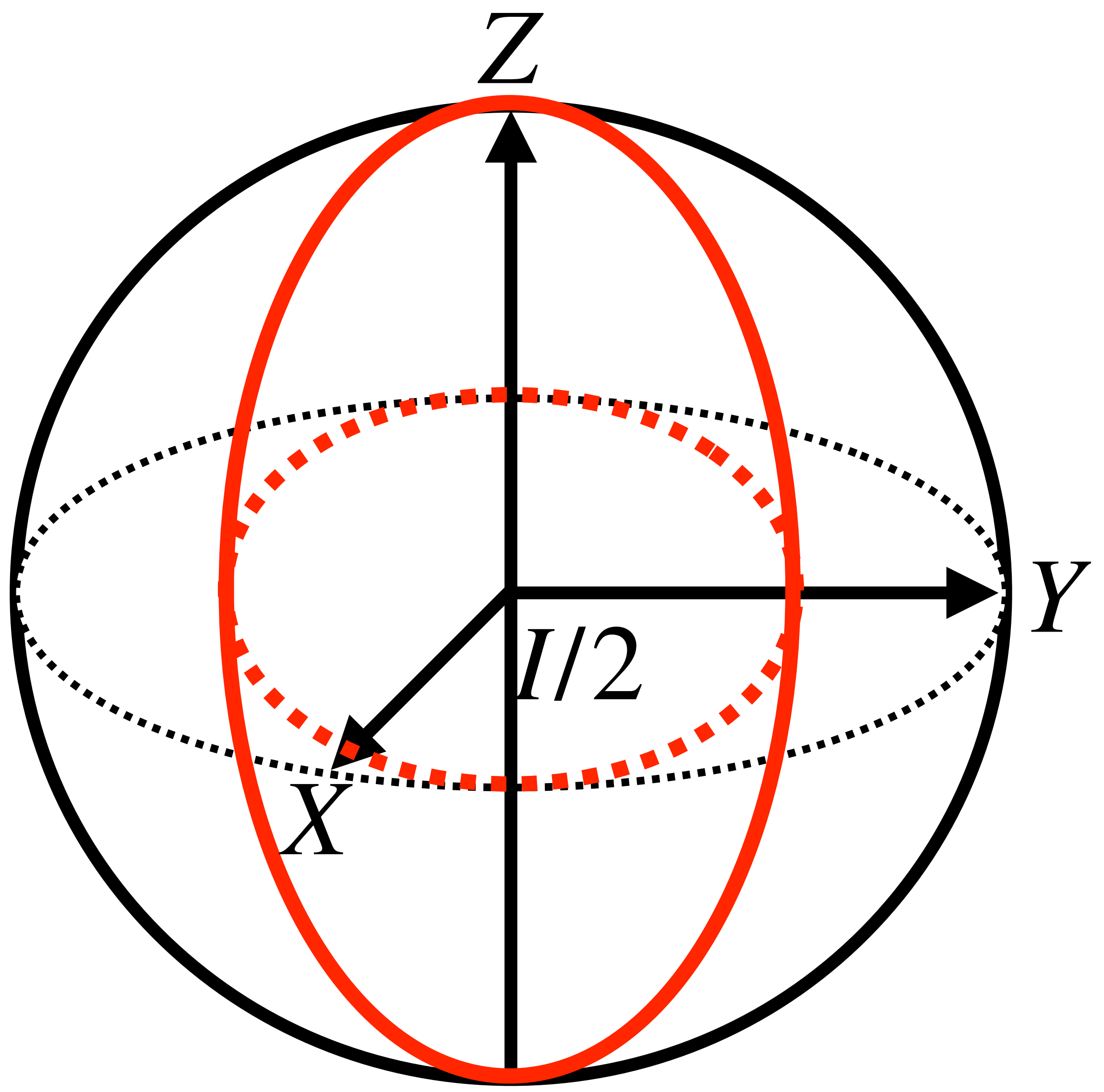
Dephasing Channel

- $\Lambda(\rho) = (1-p)\rho + pZ\rho Z$
- An example of how this map can arise:



$$\Lambda(\rho) = \int d\phi\, f(\phi) R_z(\phi)\rho R_z^{\dagger}(\phi) = \begin{bmatrix} \rho_{00} & e^{-t/T_2}\rho_{01} \\ e^{-t/T_2}\rho_{10} & \rho_{11} \end{bmatrix}$$

- $p = \dfrac{1-\exp\left(-t/T_2\right)}{2}$



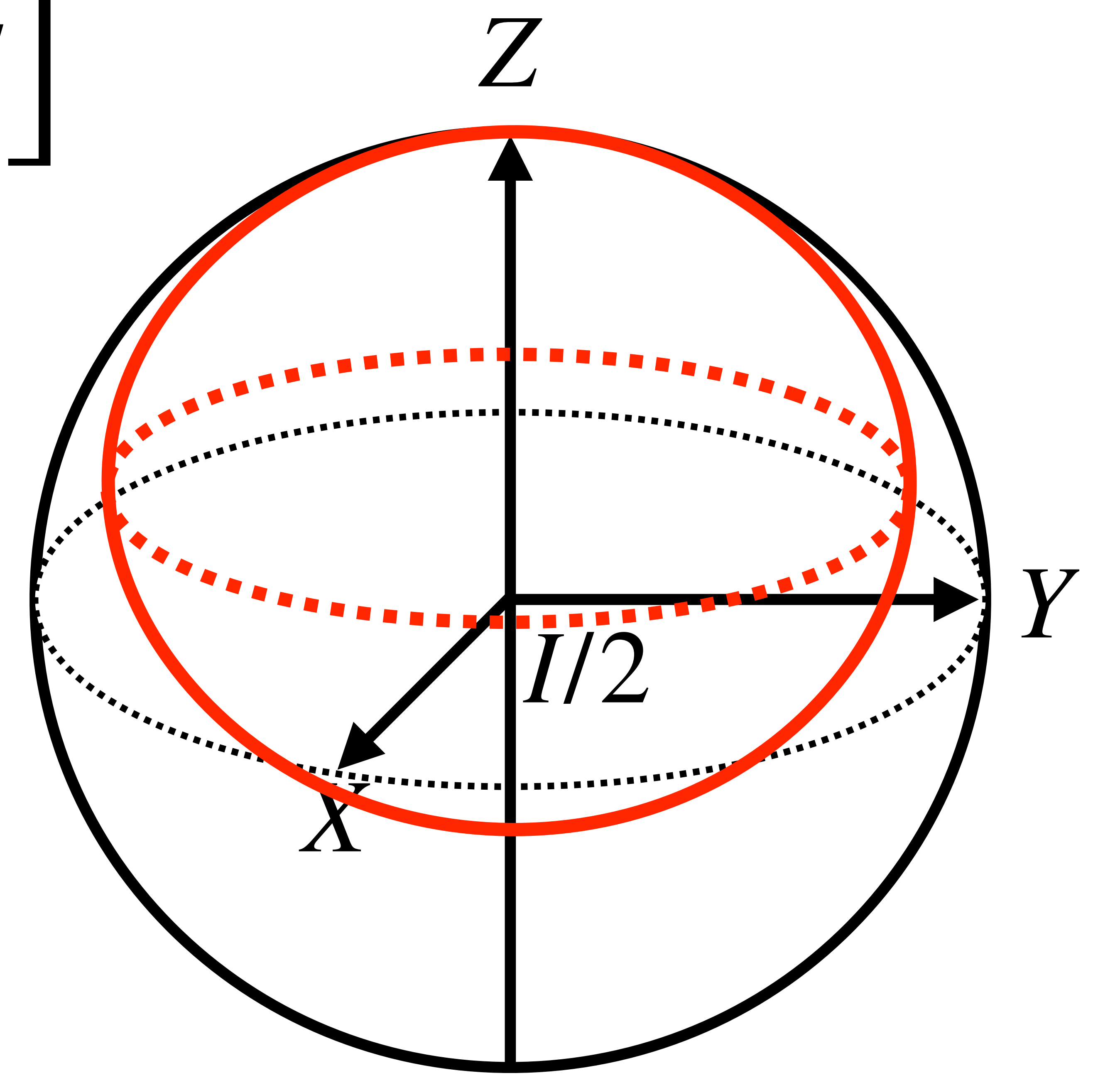
Amplitude Damping

- $\Lambda(\rho) = \begin{bmatrix} \left(\rho_{00} - \rho_{00}^{eq}\right) e^{-t/T_1} + \rho_{00}^{eq} & \rho_{01} e^{-t/2T_1} \\ \rho_{10} e^{-t/2T_1} & \left(\rho_{11} - \rho_{11}^{eq}\right) e^{-t/T_1} + \rho_{11}^{eq} \end{bmatrix}$

- For purely amplitude damping channel, $T_2 = 2T_1$.

- In general, $T_2 \leq 2T_1$.

- $\lim_{t\to\infty} \Lambda(\rho) = \rho^{eq}$ ➡ Often used for state initialization.

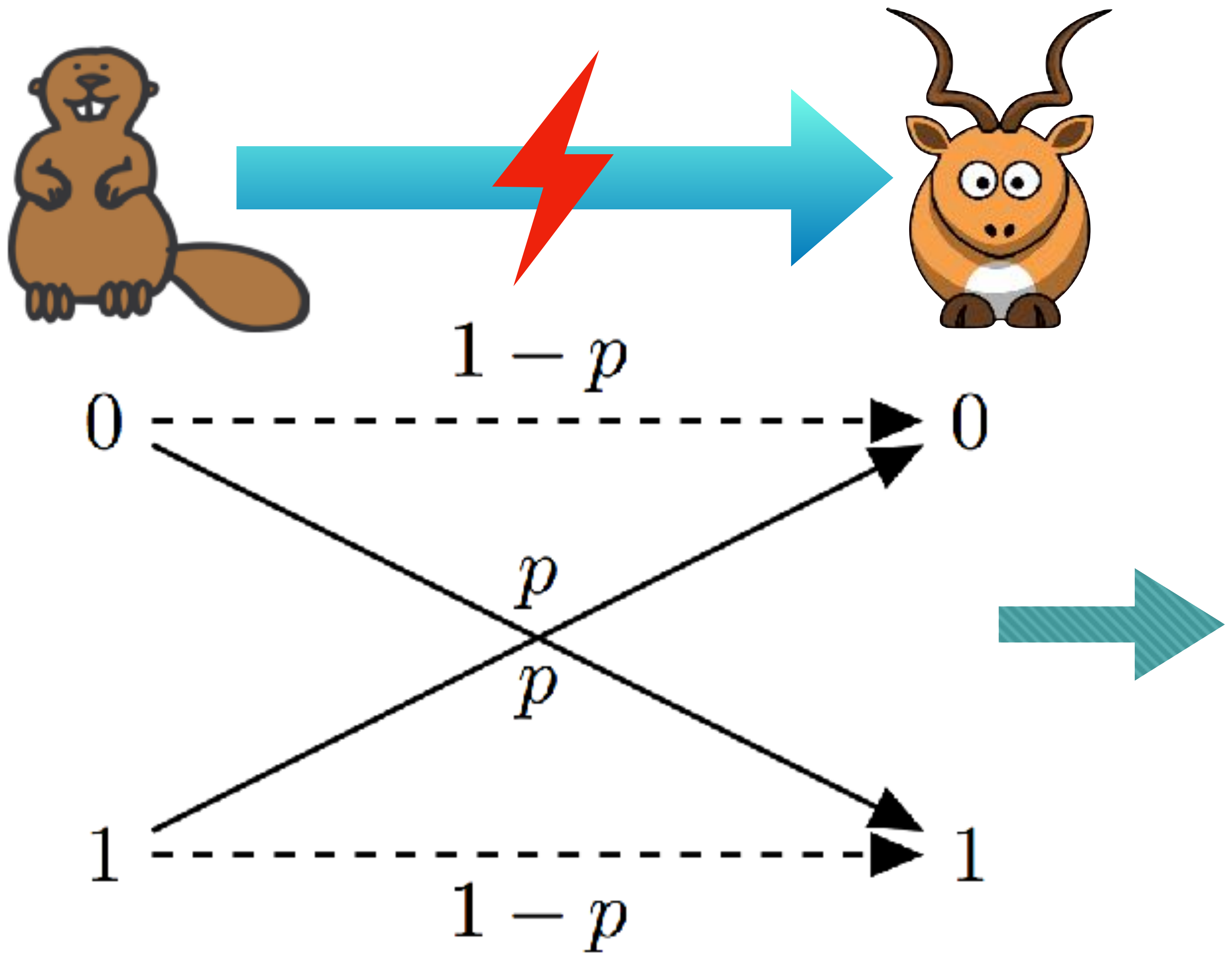


Active Quantum Error Correction

The goal of quantum error correction is to use redundancy and correction to realize logical qubits with logical error rates below the error rate of the elementary constituent qubits. .

*Recall: $\Lambda\left(\rho\right) = Tr_E\left(U\rho \otimes \sigma_E U^\dagger\right)$

Classical Repetition Code



3-bits encoding & majority vote

0→000 1→111

000, 001, 010, 100 } 000

111, 110, 101, 011 } 111

$(1-p)^3$	no error	✓	O(1)
$3p(1-p)^2$	1 error	✓	O(p)
$3(1-p)p^2$	2 errors	✗	O(p²)
p^3	3 errors	✗	O(p³)

- Probability to fail is changed from p to $3p(1-p)^2 + p^3$: Improvement as long as $p < 1/2$.
- Can add more bits (redundancy) to correct more errors.
- Quantum case is not as simple!

QEC Has To Overcome…

- Measurement destroys superposition.
- No cloning theorem prohibits repetition.
- Must correct multiple types of errors, i.e., bit-flip and phase-flip.
- Continuous errors.

Digitization of Noise

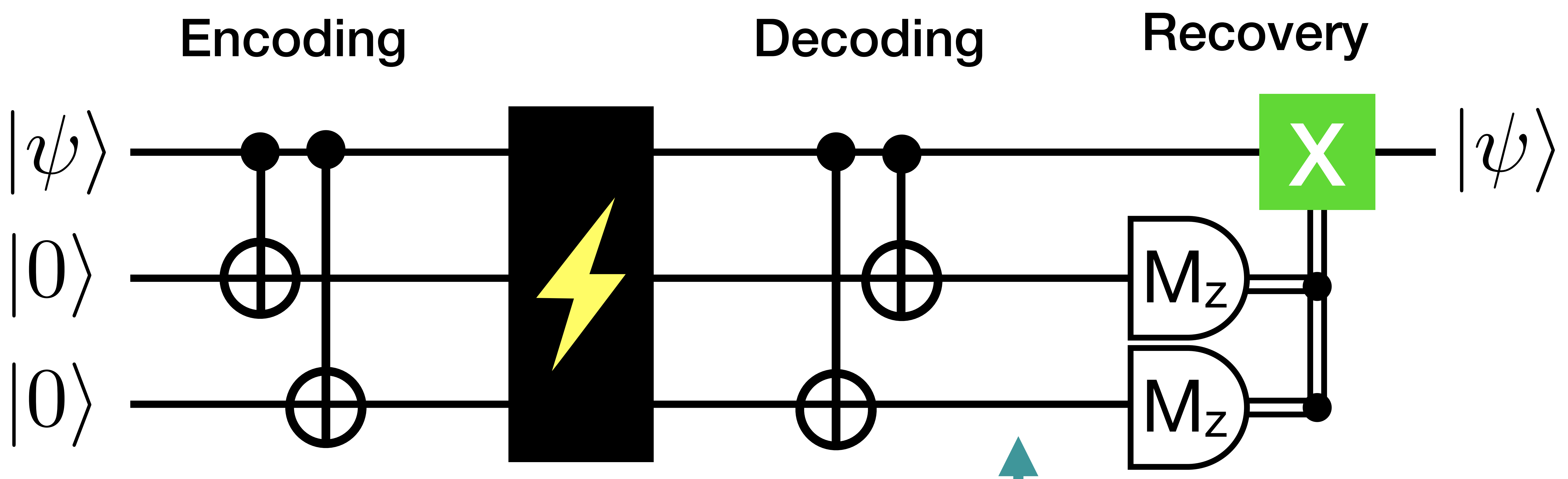
$$|0\rangle|E\rangle \to |0\rangle|E_{00}\rangle + |1\rangle|E_{01}\rangle \qquad |1\rangle|E\rangle \to |0\rangle|E_{10}\rangle + |1\rangle|E_{11}\rangle$$

$$\begin{aligned}
(\alpha|0\rangle + \beta|1\rangle)|E\rangle \to \alpha\left(|0\rangle|E_{00}\rangle + |1\rangle|E_{01}\rangle\right) + \beta\left(|0\rangle|E_{10}\rangle + |1\rangle|E_{11}\rangle\right) \\
= (\alpha|0\rangle + \beta|1\rangle) \otimes \left(|E_{00}\rangle + |E_{11}\rangle\right)/2 \\
+ (\alpha|0\rangle - \beta|1\rangle) \otimes \left(|E_{00}\rangle - |E_{11}\rangle\right)/2 \\
+ (\alpha|1\rangle + \beta|0\rangle) \otimes \left(|E_{01}\rangle + |E_{10}\rangle\right)/2 \\
+ (\alpha|1\rangle - \beta|0\rangle) \otimes \left(|E_{01}\rangle - |E_{10}\rangle\right)/2 \\
= I|\psi\rangle|E_I\rangle + Z|\psi\rangle|E_Z\rangle + X|\psi\rangle|E_X\rangle + XZ|\psi\rangle|E_{XZ}\rangle
\end{aligned}$$

Similar Pauli-expansion holds for n-qubits: $|\psi_n\rangle \otimes |E\rangle \to \sum_j \varepsilon_j |\psi_n\rangle \otimes |E_j\rangle$

Design QECC so that a subset of Pauli errors $\mathscr{E} \subseteq \{I, X, Y, Z\}^{\otimes n}$ can be detected

Smallest Code: a Bit-Flip Error



$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

↓ Encoding

$$|\psi\rangle_L = \alpha|000\rangle + \beta|111\rangle$$

Error	Resulting State
$I^{\otimes 3}$	$\alpha\,|000\rangle + \beta\,|100\rangle = |\psi\rangle \otimes |00\rangle$
X_1	$\alpha\,|111\rangle + \beta\,|011\rangle = (X\,|\psi\rangle) \otimes |11\rangle$
X_2	$\alpha\,|010\rangle + \beta\,|110\rangle = |\psi\rangle \otimes |10\rangle$
X_3	$\alpha\,|001\rangle + \beta\,|101\rangle = |\psi\rangle \otimes |01\rangle$

Error Syndrome

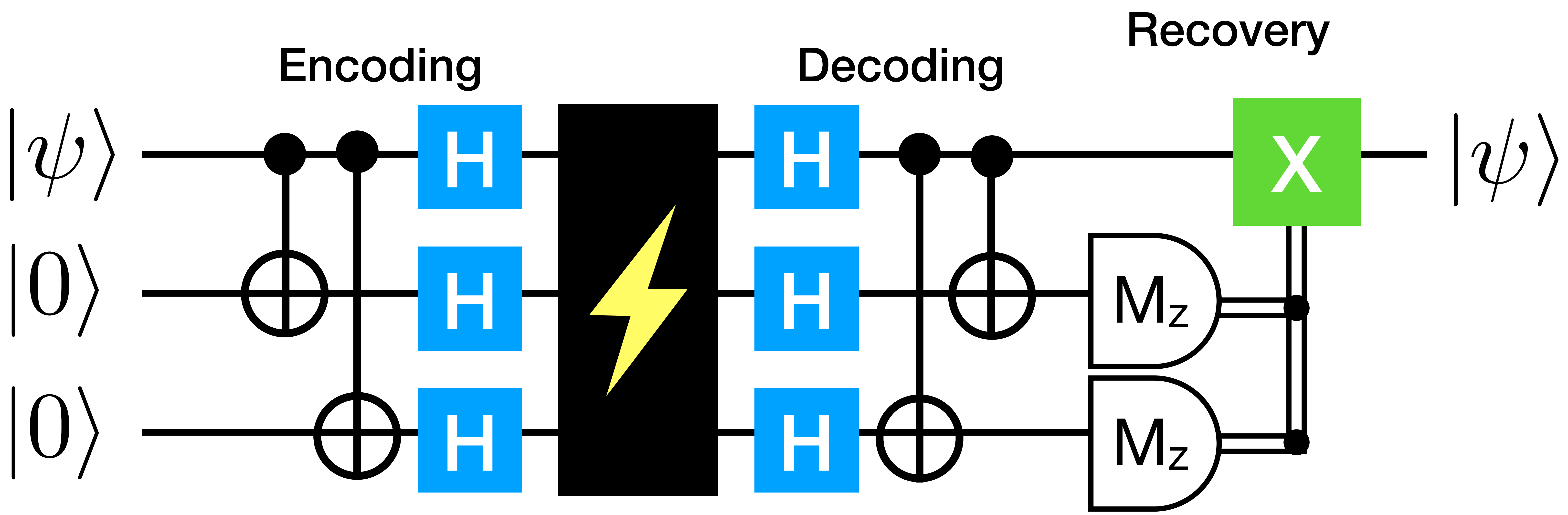
Redundancy, not copying

Measure the error, not the data

Smallest Code: a Phase-Flip Error

What about a phase-flip error? $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Since $HZH = X$, $HXH = Z$



$$|\psi\rangle_L = \alpha|{+}{+}{+}\rangle + \beta|{-}{-}{-}\rangle$$

Shor's 9-qubit Code

Corrects a Z error

$$\alpha|0\rangle + \beta|1\rangle \to \alpha|{+}{+}{+}\rangle + \beta|{-}{-}{-}\rangle$$
$$= \alpha(|0\rangle + |1\rangle)^{\otimes 3} + \beta(|0\rangle - |1\rangle)^{\otimes 3}$$
$$\to \alpha(|000\rangle + |111\rangle)^{\otimes 3} + \beta(|000\rangle - |111\rangle)^{\otimes 3}$$

Corrects an X error

- Encodes a single qubit and corrects any single-physical-qubit error.
- Code concatenation: Take the elementary qubits of the codewords of a code C, replace them by encoded qubits of a new code C'.
- Also correct $Y = -iXZ$ error (global phase irrelevant).

Small Error on Every Qubit

- What if we have a small error $U_\epsilon \approx I + \epsilon E$, $|\epsilon| \ll 1$ on every qubit?
- Then for n qubits, $U_\epsilon^{\otimes n}|\psi\rangle = |\psi\rangle + \epsilon\left(E^{(1)} + \ldots + E^{(n)}\right)|\psi\rangle + O\left(\epsilon^2\right)$.
- If the QECC corrects one-qubit errors, it corrects the sum of the $E^{(i)}$s. Therefore the state remains correct to order $O(\epsilon^2)$.
- A code correcting t errors keeps the state correct to order ϵ^{t+1}.

QEC Has To Overcome...

- Measurement destroys superposition. ☑
- No cloning theorem prohibits repetition. ☑
- Must correct multiple types of errors, i.e., bit-flip and phase-flip. ☑
- Continuous errors. ☑

Fault-Tolerance

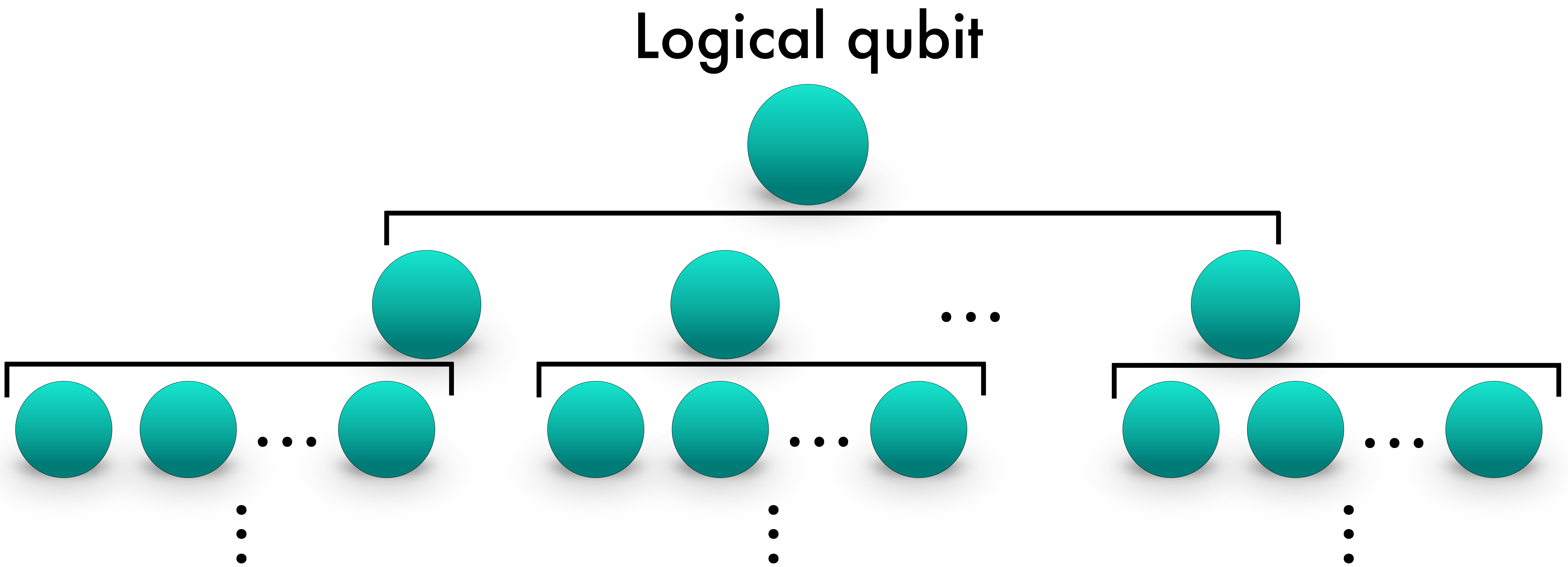
- Any realization will suffer from imperfections. There is no guarantee that QEC can help as it may introduce more errors than it takes away.
- The Theory of Fault-tolerance comes to rescue!

Desiderata

- Fault-tolerant error correction
- Fault-tolerant state preparation
- Fault-tolerant computation (e.g., universal set of gate)
- Fault-tolerant measurement

Thus, things get pretty complicated...

Concatenated codes

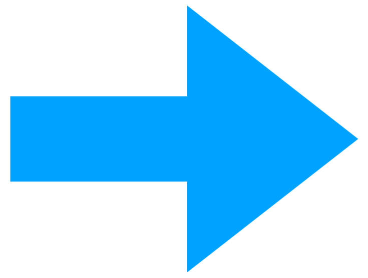


level-1: $p \to cp^2$

level-2: $cp \to c(cp^2)^2$

$\vdots$

level-k: $p_k \to (cp)^{2^k}/c$



- Concatenation uses exponentially increasing amount of resources, but improves the error rate double-exponentially as long as $p < 1/c$
- We're ready to see the Threshold theorem.

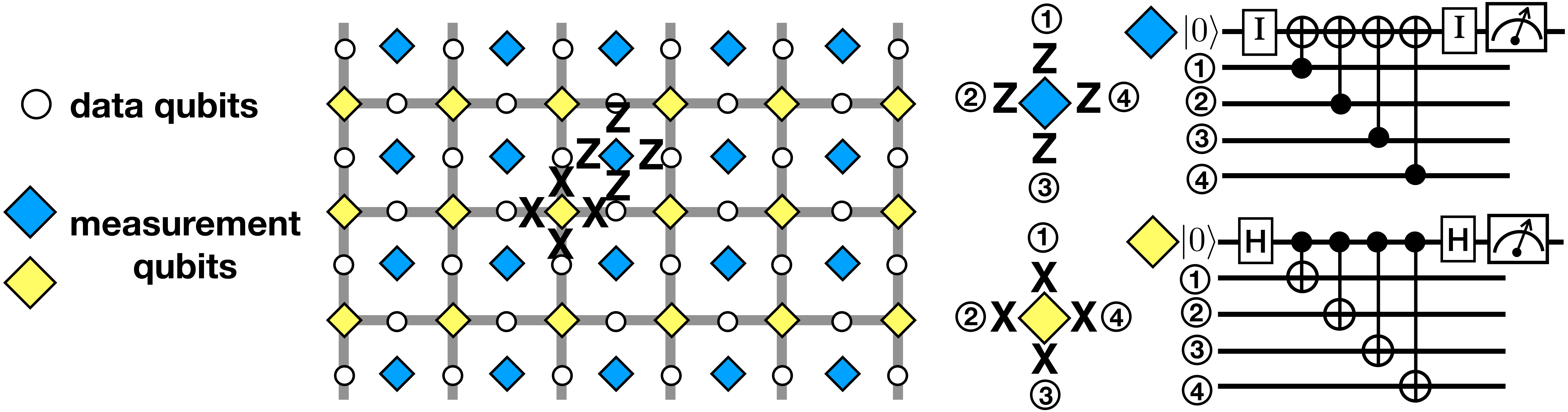
Fault-Tolerance Threshold Theorem

- There exists a threshold error probability p_t such that, if the error rate per gate and time step is $p < p_t$, arbitrarily long quantum computations are possible.

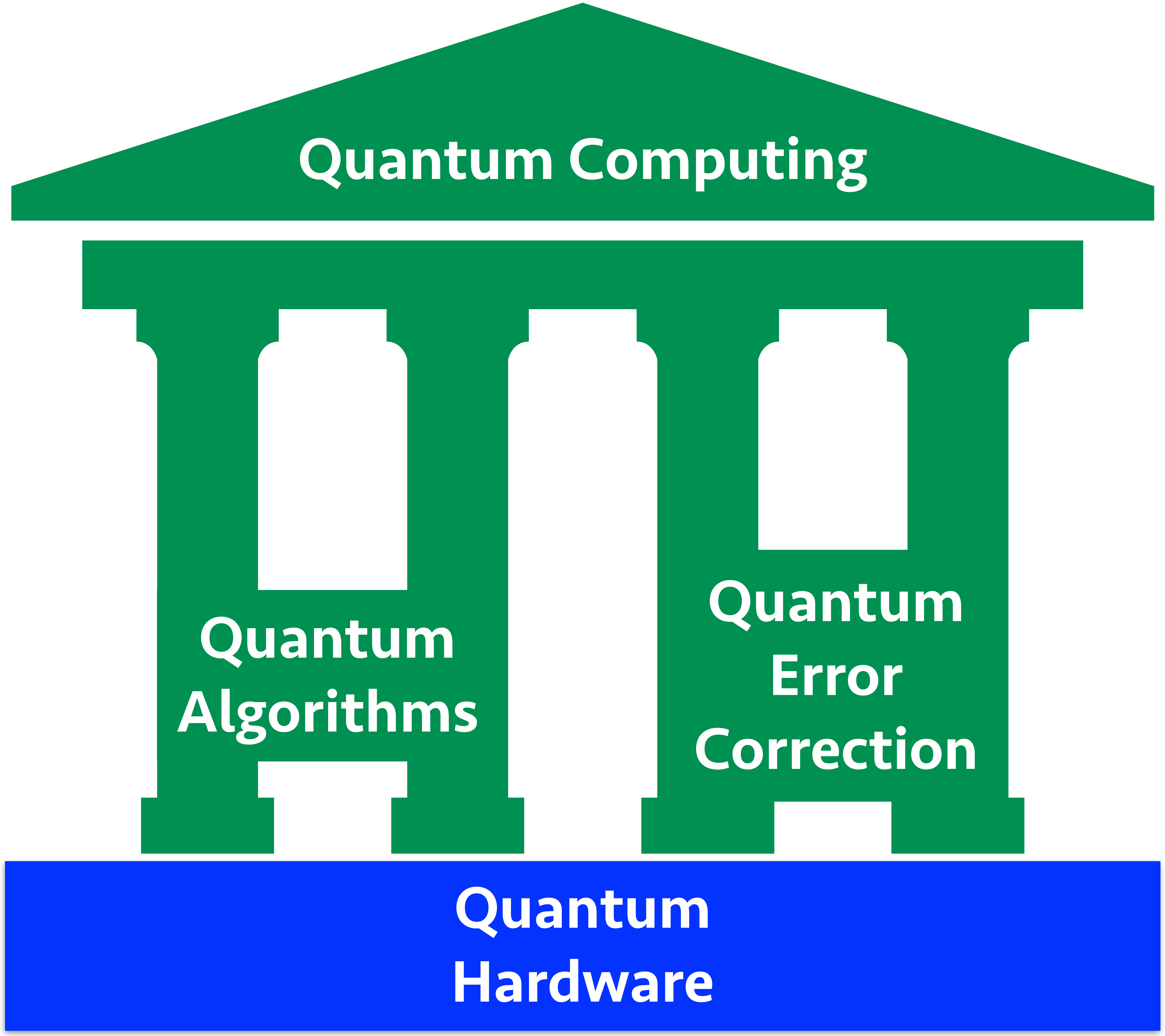
- More precisely, a quantum circuit of size N can be simulated with a probability of final error at most ϵ using $O(poly(\log(N/\epsilon))N)$ gates whose components fail with probability at most $p < p_t$, given reasonable assumptions about the underlying hardware.

Remaining Challenges

- Concatenated codes face some practical issues, such as:
  - Require long distance interaction among qubits.
  - A level of concatenation, increases the resource exponentially.
- Topological QEC codes (e.g. surface code) can solve the above issues.



Theoretical Pillars for Quantum Computing

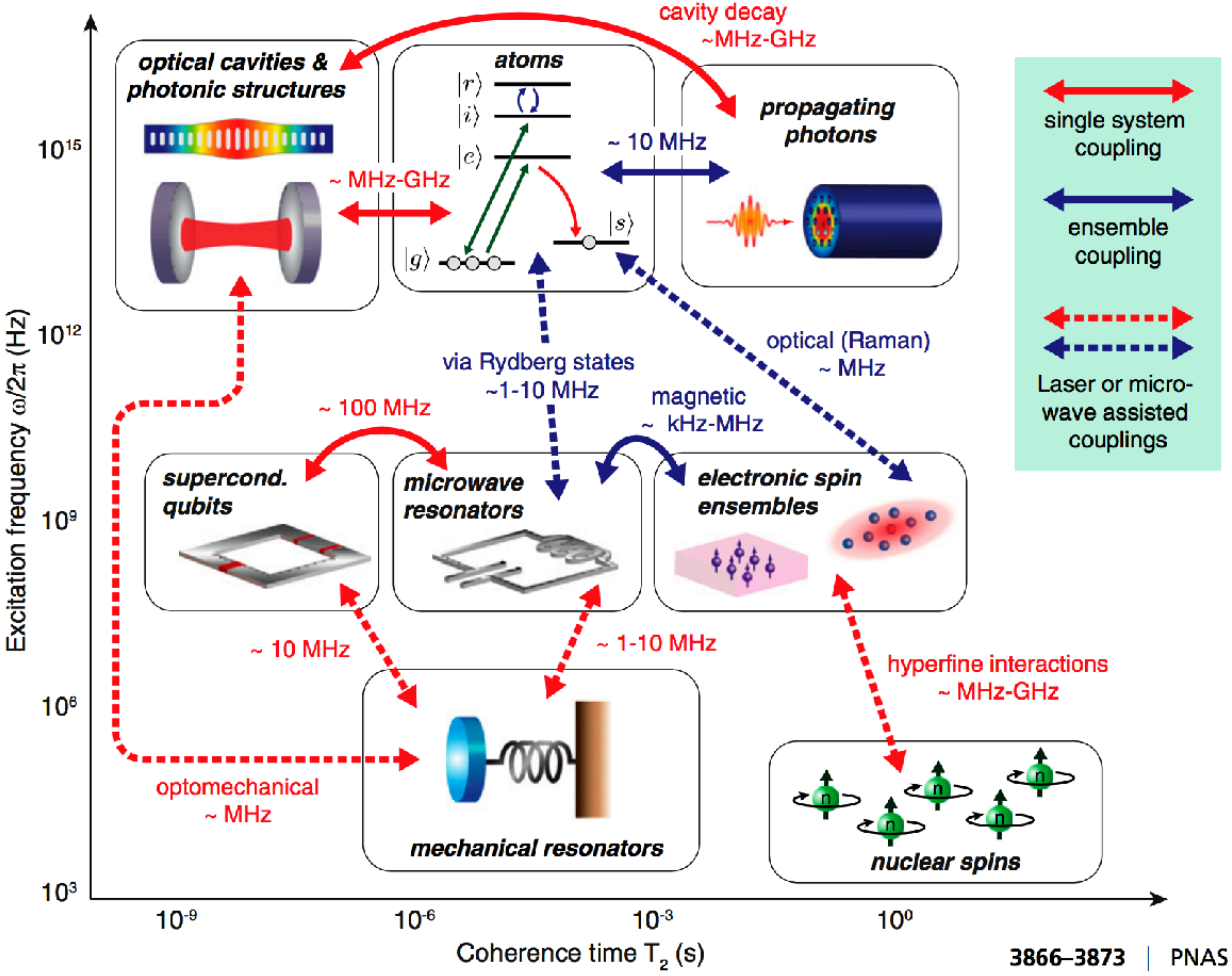


The Physical Implementation of QC - DiVincenzo's Criteria

- A scalable physical system with well characterized qubit
- The ability to initialize the state of qubit to a simple fiducial state
- Relevant decoherence times >> gate operation times
- A universal set of quantum gates
- A qubit-specific measurement capability
- The ability to interconvert stationary and flying qubits
- The ability to faithfully transmit flying qubits between specified locations

arXiv:quant-ph/0002077v3 (2000)

Physical Systems for Quantum Computing



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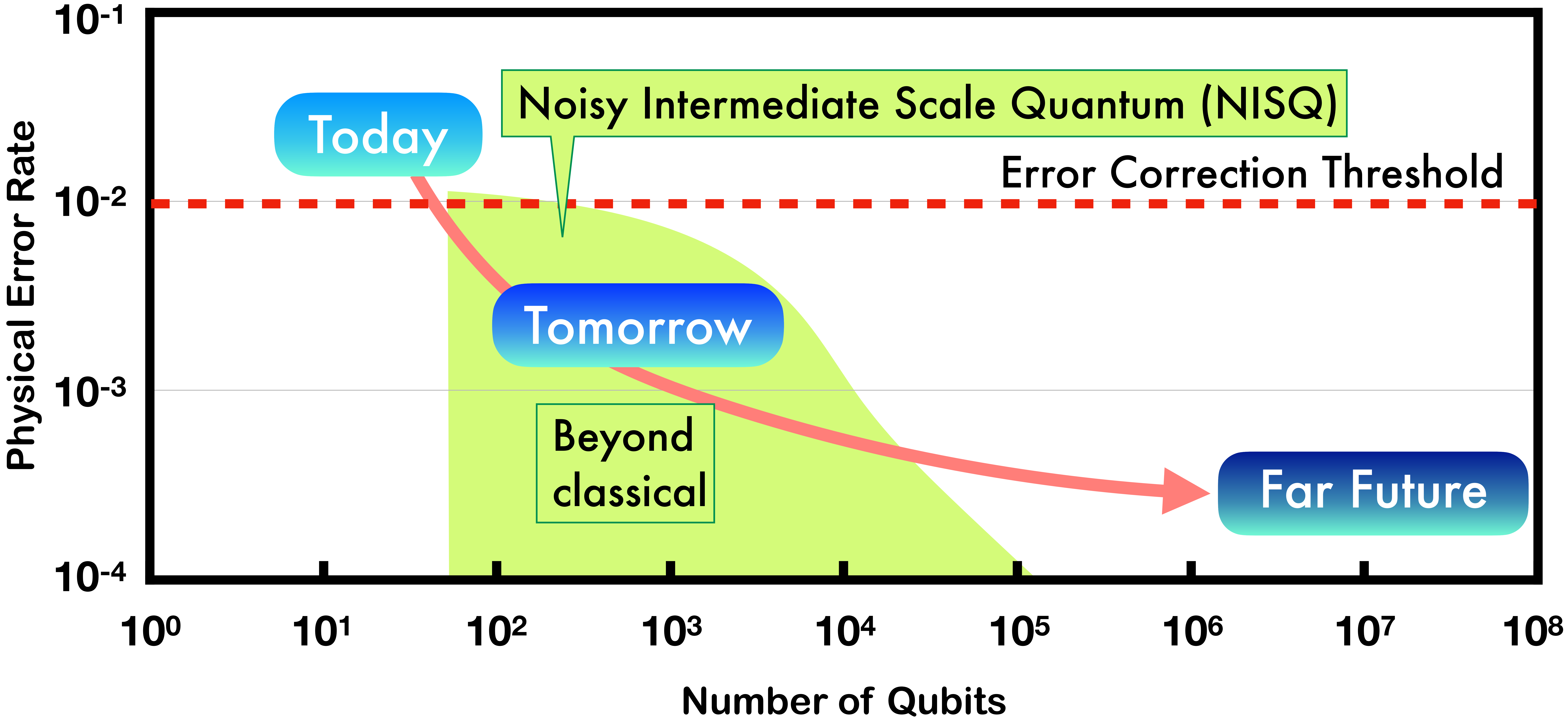
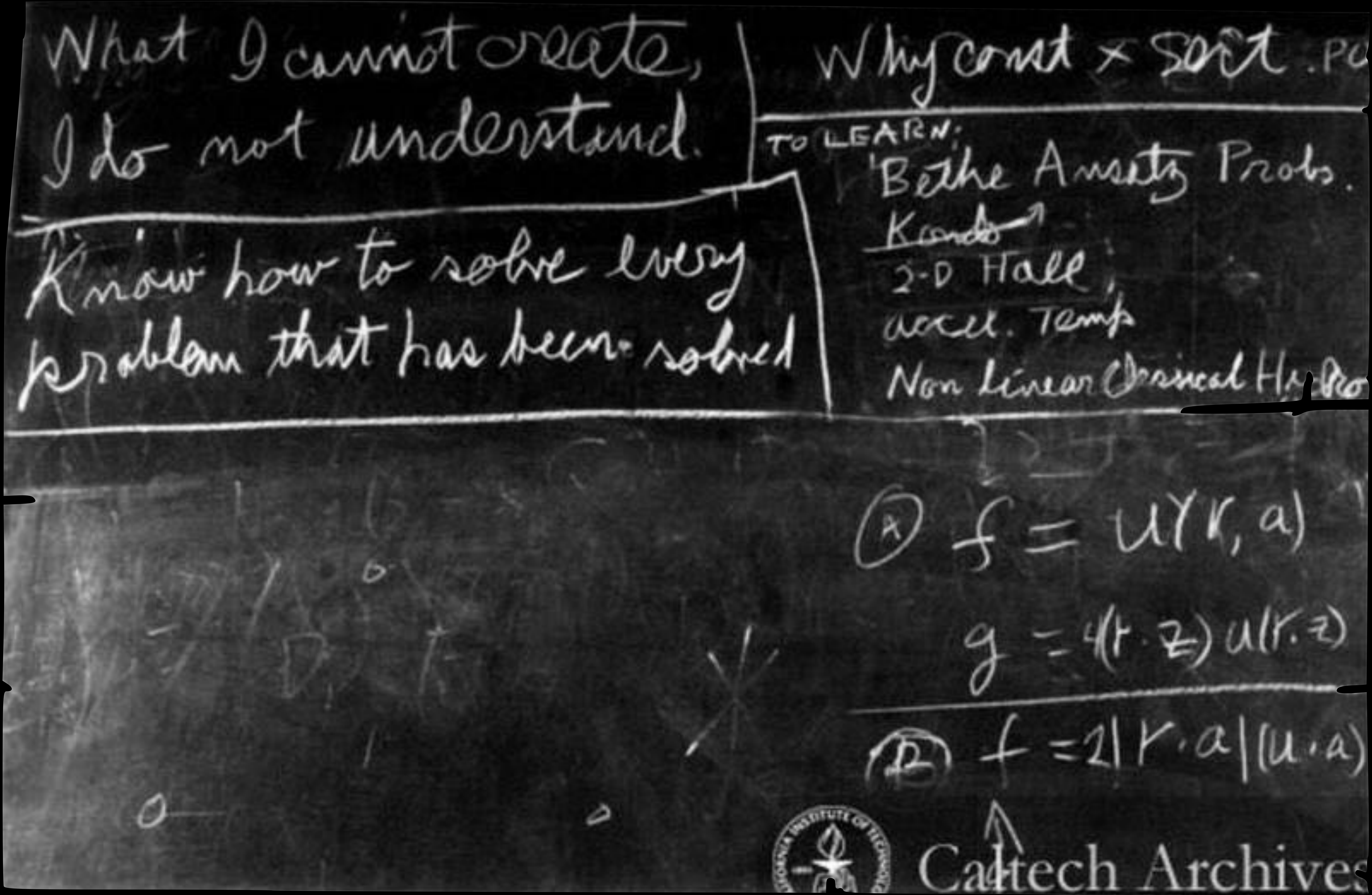
Quantum Hardware Roadmap




What I cannot create,
I do not understand.
Why const × sort. pc
TO LEARN:
Bethe Ansatz Probs.
Kondo
2-D Hall
accel. Temp
Non Linear Classical Hydro
Know how to solve every
problem that has been solved
(A) f = u r (r, a)
g = 4(r · z) u(r · z)
(B) f = 2 |r · a| (u · a)
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