19. October: Green's function: properties, construction.

Problem: we seek solutions of second-order non-homogeneous ordinary differential equation using a method known as the method of Green's function

Let's consider the following problem:

Find the solution of the Non-homogeneous equation

$$L_x u(x) = f(x)$$

$$B_1[u] = \alpha_1 u(a) + \beta_1 \frac{du}{dx}\Big|_{x=a} + \gamma_1 u(b) + \delta_1 \frac{du}{dx}\Big|_{x=b} = 0$$

$$B_2[u] = \alpha_2 u(a) + \beta_2 \frac{du}{dx}\Big|_{x=a} + \gamma_2 u(b) + \delta_2 \frac{du}{dx}\Big|_{x=b} = 0$$

B_1, B_2 are homogeneous boundary conditions.

Our problem is equivalent to the algebraic problem of finding solutions of the operator equation

$$L|u\rangle = |f\rangle, \text{ where}$$

L is defined by L_x together with the boundary conditions $B_1[u]$, $B_2[u]$.

Provided there exists an operator G, satisfying

$$LG = E, \text{ where } E \equiv \text{identity operator},$$

the solution of $L|u\rangle = |f\rangle$ is $|u\rangle = G|f\rangle$.

If the operator G does exist, it is called Green's operator.

Let us tentatively write G as an integral operator.

$$G = \int_a^b \int_a^b dx'\, dx'' \, |x'\rangle\, w(x')\, G(x', x'')\, w(x'') \langle x''|$$

$$\langle x | L G | y\rangle = L_x \langle x | G | y\rangle$$

question: are we assuming $LG = GL = E$?

Response: No! only $LG = E$.

$$\langle x | G | y\rangle = \int_a^b \int_a^b dx'\, dx'' \underbrace{\langle x | x'\rangle}_{\frac{\delta(x-x')}{w(x')}} w(x')\, G(x', x'')\, w(x'') \underbrace{\langle x'' | y\rangle}_{\frac{\delta(x''-y)}{w(x'')}}$$

$$= \int_a^b \int_a^b dx'\, dx'' \, \delta(x-x')\, G(x', x'')\, \delta(x''-y)$$

$$\langle x|G|y\rangle = \int_a^b dx'\,\delta(x-x')\underbrace{\int_a^b dx''\,G(x',x'')\,\delta(x''-y)}_{G(x',y)}$$

$$= \underbrace{\int_a^b dx'\,\delta(x-x')\,G(x',y)}_{G(x,y)}$$

$$\langle x|G|y\rangle = G(x,y)$$

$$\left.\begin{aligned}\langle x|\underbrace{LG}_{E}|y\rangle &= L_x\,\langle x|G|y\rangle = L_x\,G(x,y)\\ &= \langle x|E|y\rangle = \langle x|y\rangle = \frac{\delta(x-y)}{w(x)}\end{aligned}\right\}\; L_x\,G(x,y) = \frac{\delta(x-y)}{w(x)}$$

Therefore. The kernel $G(x,y)$ of the operator G satisfies $L_x G(x,y) = \frac{\delta(x-y)}{w(x)}$

(1) $L_x G(x,y) = \frac{\delta(x-y)}{w(x)} \Rightarrow L G(x,y) = 0$ for $x \neq y$

that means that $G(x,y)$ satisfies the [HE]

$L_x G(x,y) = \frac{\delta(x-y)}{w(x)}$ satisfies the same equation as

$u(x)$, $[L_x u(x) = f(x)]$, except that the non-homogeneous

term $f(x)$ has been replaced by $\frac{\delta(x-y)}{w(x)}$.

(2) $G(x,y)$ as a function of x must satisfy the same homogeneous boundary conditions as those imposed on $u(x)$.

$\langle x | L G | y \rangle$ is meaningful when $G|y\rangle$ belongs to the domain of L

NB. Whenever a solution of $L_x G(x,y) = \frac{\delta(x-y)}{w(x)}$ exists,
the integral operator G is well defined
and is the right inverse of the differential operator L.

Properties of $G(x,y)$.

① $G(x,y) = \overline{G(y,x)}$ for Hermitian operator L
If the coefficients of L_x are real, $G(x,y) = G(y,x)$ (Symmetric)

② $G(x,y)$ is continuous at $x=y$

③ $\frac{\partial G(x,y)}{\partial x}$ is discontinuous at $x=y$ and the jump of discontinuity at y is given by $\lim_{\substack{x \to y \\ x > y}} \frac{\partial G(x,y)}{\partial x} - \lim_{\substack{x \to y \\ x < y}} \frac{\partial G(x,y)}{\partial x} = \frac{1}{w(y)\,a(y)}$

How do we proceed concretely?

Solve $\begin{cases} a(x)\dfrac{d^2u(x)}{dx^2} + b(x)\dfrac{du(x)}{dx} + c(x)u(x) = f(x), & x \in [a,b] \\ & a(x) \neq 0. \\ \left.\begin{array}{l} B_1[u] = 0 \\ B_2[u] = 0 \end{array}\right\} \text{Homogeneous boundary conditions.} \end{cases}$

Here $L_x = a(x)\dfrac{d^2}{dx^2} + b(x)\dfrac{d}{dx} + c(x)$

[HE] : $a(x)\dfrac{d^2u(x)}{dx^2} + b(x)\dfrac{du(x)}{dx} + c(x)u(x) = 0$

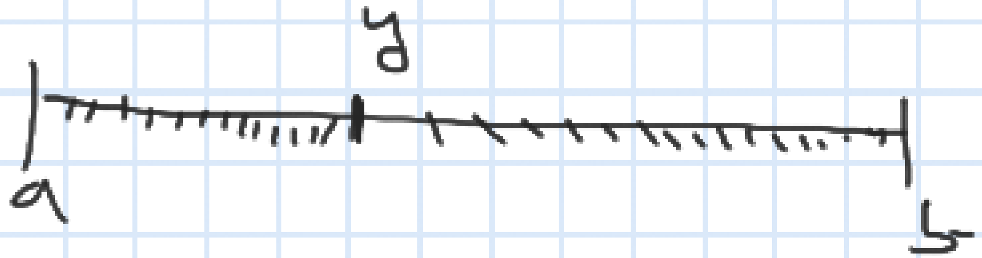
Suppose $G(x,y)$ is the associated Green's function

$L_x G(x,y) = \dfrac{\delta(x-y)}{w(x)} \Rightarrow L_x G(x,y) = 0$, for $x \neq y \Rightarrow$

Except $x = y$, $G(x,y)$ satisfies the HE on $[a,b] \setminus \{y\}$

$$\begin{cases} a(x)\dfrac{\partial^2 G(x,y)}{\partial x} + b(x)\dfrac{\partial G}{\partial x}(x,y) + c(x)\,G(x,y) = 0 \text{ for } x \neq y \\ B_1[G] = 0 \\ B_2[G] = 0 \end{cases}$$

Suppose that $u_1(x)$, $u_2(x)$ are a fundamental set of solutions of HE, we can express the most general solution of $L_x G(x,y) = 0$, only valid on $[a,b]$ to the left and to the right at the point $x = y$



$$G(x,y) = c_1 u_1(x) + c_2 u_2(x) \text{ for } a \leq x < y$$

$$G(x,y) = d_1 u_1(x) + d_2 u_2(x) \text{ for } y < x \leq b.$$

where c_1, c_2, d_1, d_2 are arbitrary constants that may depend on y.

$$G(x,y) = \begin{cases} c_1(y)\, u_1(x) + c_2(y)\, u_2(x) & a \le x < y \\ d_1(y)\, u_1(x) + d_2(y)\, u_2(x) & y < x \le b. \end{cases}$$

Let's determine $c_1(y)$, $c_2(y)$, $d_1(y)$, $d_2(y)$

(1) $B_1[G(x,y)] = 0$

$B_2[G(x,y)] = 0$

(2) $G(x,y)$ must be continuous at $x=y$ $\lim\limits_{\substack{x\to y \\ x<y}} G(x,y) = \lim\limits_{\substack{x\to y \\ x>y}} G(x,y)$

(3) $\lim\limits_{\substack{x\to y \\ x>y}} \dfrac{\partial G(x,y)}{\partial x} - \lim\limits_{\substack{x\to y \\ x<y}} \dfrac{\partial G(x,y)}{\partial x} = \dfrac{1}{w(y)\, a(y)}$

These 3 conditions imposed on $G(x,y)$ determined the four constants $c_1(y), c_2(y), d_1(y), d_2(y)$ and hence the Green's function.

once we have the Green's function $G(x,y)$, the solution of

$$\begin{cases} L_x u(x) = f(x), & x \in [a,b] \\ B_1\{u\} = 0 \\ B_2\{u\} = 0 \end{cases}$$

is

$$u(x) = \int_a^b dy\, w(y)\, G(x,y)\, f(y).$$

$L|u\rangle = |f\rangle$

if G exists: $LG = E$. $\Rightarrow$ $|u\rangle = G|f\rangle$

$$\langle x|u\rangle = \langle x|G|f\rangle$$

$$u(x) = \int_a^b dy\, w(y)\, G(x,y)\, f(y).$$