12. october 2022 : session 2.

Integral and differential operators

(1) Notations

- The Dirac delta function.
- vector $|f\rangle$ and its components $f(x)$, $a \leq x \leq b$.

(2) Integral operator

(3) Formal differential operator

(1) Notations

(a) The Dirac delta function.

The Dirac delta function is a useful mathematical object that has various physical backgrounds in many different disciplines.

There are many ways to actually define the Dirac delta function. Let's introduce the following definition

If a function $f(x)$ satisfies the two conditions

C_1) $f(x) = \begin{cases} +\infty & \text{for } x = x_0 \\ 0 & \text{otherwise } (x \neq x_0) \end{cases}$

C_2) $\int_{-\infty}^{+\infty} f(x)\, dx = 1$

Then $f(x)$ is called the Dirac delta function and denoted $f(x) = \delta(x - x_0)$

NB: It is usually called Dirac function for simplicity.

These are the main properties of the Dirac delta function that we need to be aware of.

$$\delta(x-x_0) = 0 \quad \text{if } x \neq x_0$$

$$\int_{-\infty}^{+\infty} \delta(x-x_0)\,dx = 1$$

$$\int_{-\infty}^{+\infty} \delta(x-x_0)\, f(x)\,dx = f(x_0)$$

$$\int_a^b \delta(x-x_0)\,dx = \begin{cases} 1 & a < x_0 < b \\ 0 & \text{otherwise} \end{cases}$$

5) vector $|f\rangle$ and its components $f(x)$.

Let's consider the entire set of values of a function $f(x)$ as representing a vector $|f\rangle$, in other words, the numbers $f(x)$ are treated as the components with "index x" of an abstract vector $|f\rangle$.

the vector $|f\rangle$ belongs to an abstract function space F which is defined as the linear vector space whose vectors are represented by functions defined on an interval $[a, b]$.

F, $|f\rangle \in F$, $|f\rangle = \{ f(x), a \leq x \leq b \}$.

Since the components of a vector are defined with respect to some basis, we formally introduce such a basis.

Such basis consist of vectors $|x\rangle$, where the index "x" that labels these vectors is continuous $(a \le x \le b)$

$|x\rangle$ and $|x'\rangle$ are orthogonal $\langle x'|x\rangle = 0$
we cannot have the normalization condition $\langle x|x\rangle = 1$
since "x" is continuous.

we set by definition $f(x) \underset{def}{=} \langle x|f\rangle$

with respect to the basis $\{|x\rangle, a \le x \le b\}$

$|f\rangle = \int_a^b dx\, w(x) f(x) |x\rangle$, where $w(x)$ is called the weight function

Let's check the expression $\langle x'|x\rangle$ and why we cannot have the normalization condition $\langle x'|x\rangle = 1$

- Let's start again with $|f\rangle = \int_a^b dx\, w(x) f(x) |x\rangle$.
- Let's multiply from the left by $\langle x'|$

$$\underbrace{\langle x'|f\rangle}_{\displaystyle \| \atop \displaystyle f(x')} = \int_a^b dx\, w(x) f(x) \langle x'|x\rangle$$

$$f(x') = \int_a^b dx\, w(x) f(x) \langle x'|x\rangle \Rightarrow$$

$w(x)\langle x'|x\rangle$ has the property of the δ-distribution and we cannot have $\langle x'|x\rangle = 1$

$$\langle x'|x\rangle = \frac{1}{\sqrt{w(x)\,w(x')}}\,\delta(x-x')$$
$$= \frac{1}{w(x)}\,\delta(x-x')$$
$$= \frac{1}{w(x')}\,\delta(x-x')$$

Here we treat $\delta(x-x'$ as a formal generalization of the Kronecker delta.

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

- Scalar product

$$\langle f|g\rangle = \int_a^b w(x)\,\bar{f}(x)\,g(x)\,dx\,, \quad w(x) \equiv \text{weight function}$$

- Identity operator

In a function space, the identity operator can be written in the form $\mathbb{I} = \int_a^b dx\, |x\rangle\, w(x)\, \langle x|$

(2) Integral operator:

We call an operator an integral operator if, using the notations we have just introduced, it can be written formally as a double integral

$$K = \int_a^b \int_a^b dx''\, dx'\, |x''\rangle\, w(x'')\, K(x'', x')\, w(x')\, \langle x'|$$

$|f\rangle \in F$.

$$|g\rangle = K|f\rangle = \int_a^b\int_a^b dx''\, dx'\, |x''\rangle\, w(x'')\, K(x'', x')\, w(x') \underbrace{\langle x'|f\rangle}_{f(x')}$$

$$\underset{\substack{\Downarrow \\ g(x)}}{\langle x|g\rangle} = \int_a^b\int_a^b dx''\, dx'\, \underbrace{\langle x|x''\rangle}_{\frac{\delta(x-x'')}{w(x'')}}\, w(x'')\, K(x'', x')\, w(x')\, g(x')$$

$$g(x) = \int_a^b dx'\, w(x')\, f(x') \int_a^b dx''\, \frac{\delta(x-x'')}{\cancel{w(x'')}}\, \cancel{w(x'')}\, K(x'', x')$$

$$g(x) = \int_a^b dx'\, w(x')\, f(x') \underbrace{\int_a^b dx''\, \delta(x-x'')\, K(x'', x')}_{K(x,\, x')}$$

$$\Downarrow$$

$$|g\rangle = K|f\rangle, \quad g(x) = \int_a^b dx'\, w(x')\, K(x, x')\, f(x')$$

$|g\rangle = K|f\rangle$ has its analytical counterpart that is $g(x) = \int_a^b dx'\, w(x')\, K(x, x')\, f(x')$ and has the form of an integral representation for the function $g(x)$ and justify the calling $K(x, x')$, the kernel of the operator K.

(3) Formal differential operator.

A linear differential operator L is an operator in a function space whose action on vectors $|f\rangle$ of this space is represented by differential form

$$\langle x|L|f\rangle = a_0(x) f(x) + a_1(x)\frac{d f(x)}{dx} + a_2(x)\frac{d^2 f(x)}{dx^2} + a_N(x)\frac{d^N f(x)}{dx^N}$$

it can be written as $\langle x|L|f\rangle = L_x f(x)$, where

$$L_x = a_0(x) + a_1(x)\frac{d}{dx} + a_2(x)\frac{d^2}{dx^2} + \cdots a_N(x)\frac{d^N}{dx^N}$$

- The symbolic expression L_x represents the differential operator L defined in an abstract function space in the sense that $L_x f(x)$ represent the vector $L|f\rangle$.
- The differential form $L_x f(x)$ is meaningful for any sufficiently differential function $f(x)$.
- The vector $L|f\rangle$ is meaningful for vectors $|f\rangle$ that belongs to a specific function space, called the domain of the operator L
- The functions $f(x)$ that represent the vectors $|f\rangle$ of the domain L must be sufficiently differentiable and also satisfy certain additional conditions called the boundary conditions

- Every linear differential equation involves some differential form of the type $L_x f(x)$, where L_x has a purely formal character
- Every expression of the type $L_x = a_0(x) + a_1(x)\frac{d}{dx} + \cdots + a_N(x)\frac{d^N}{dx^N}$ is called a formal differential operator.
- We are interested here in the formal differential operator of second order

$$L_x = a_0(x) + a_1(x)\frac{d}{dx} + a_2(x)\frac{d^2}{dx^2}$$

A differential operator L may have an inverse operator L^{-1} which is bounded and completely continuous.

The inverse of a linear differential operator is an integral operator, whose kernel is called the Green's function of the differential operator.