Trace decreasing dynamical maps: Entanglement dynamics and quantum communication rate

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Two-Day Workshop to celebrate the 60th anniversary of the paper on dynamical maps by E. C. G. Sudarshan, P. M. Mathews, and J. Rau

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Stochastic Dynamics of Quantum-Mechanical Systems

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(Received August 15, 1960)

The most general dynamical law for a quantum mechanical system with a finite number of levels is formulated. A fundamental role is played by the so-called "dynamical matrix" whose properties are stated in a sequence of theorems. A necessary and sufficient criterion for distinguishing dynamical matrices corresponding to a Hamiltonian time-dependence is formulated. The non-Hamiltonian case is discussed in detail and the application to paramagnetic relaxation is outlined.

$$A_{sr,s'r'} = (A_{rs,r's'})^*, \tag{11'}$$

$$x_r^* x_s A_{rs,r's'} y_{r'} y_{s'}^* \geqslant 0, \tag{12'}$$

$$A_{rr,r's'} = \delta_{r's'} \tag{13'}$$

which give fairly complicated properties for the A matrix. To display these properties in a more transparent fashion, as well as for further development, it is advantageous to introduce another $n^2 \times n^2$ matrix B related to A and defined by

$$B_{rr',ss'} = A_{rs,r's'}. \tag{14}$$

It immediately follows that B is Hermitian and positive semidefinite; we can rewrite (11′) and (12′) in the form:

$$B_{rr',ss'} = (B_{ss',rr'})^*, \quad \text{(Hermiticity)} \tag{15}$$

$$z_{rr'}^* B_{rr',ss'} z_{ss'} \geqslant 0. \quad \text{(positive semidefiniteness)} \tag{16}$$

The trace condition (13′) is still complicated and becomes

$$B_{rr',rs'} = \delta_{r's'}; \tag{17}$$

$$\Lambda^\dagger[I] = I$$

by summing with respect to the other indices also, we obtain the weaker statement

$$B_{rr',rr'} = \delta_{r'r'} = n. \tag{18}$$

$$A_{sr,s'r'} = (A_{rs,r's'})^*, \tag{11'}$$

$$x_r^* x_s A_{rs,r's'} y_{r'} y_{s'}^* \geqslant 0, \tag{12'}$$

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SCIENTIFIC REPORTS

Corrected: Author Correction

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All-optical implementation of collision-based evolutions of open quantum systems

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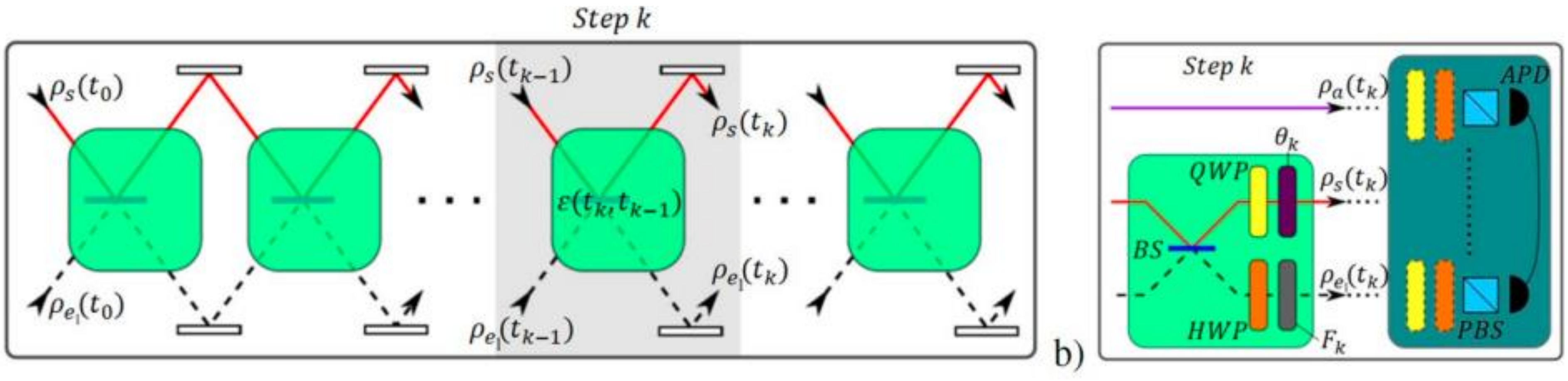
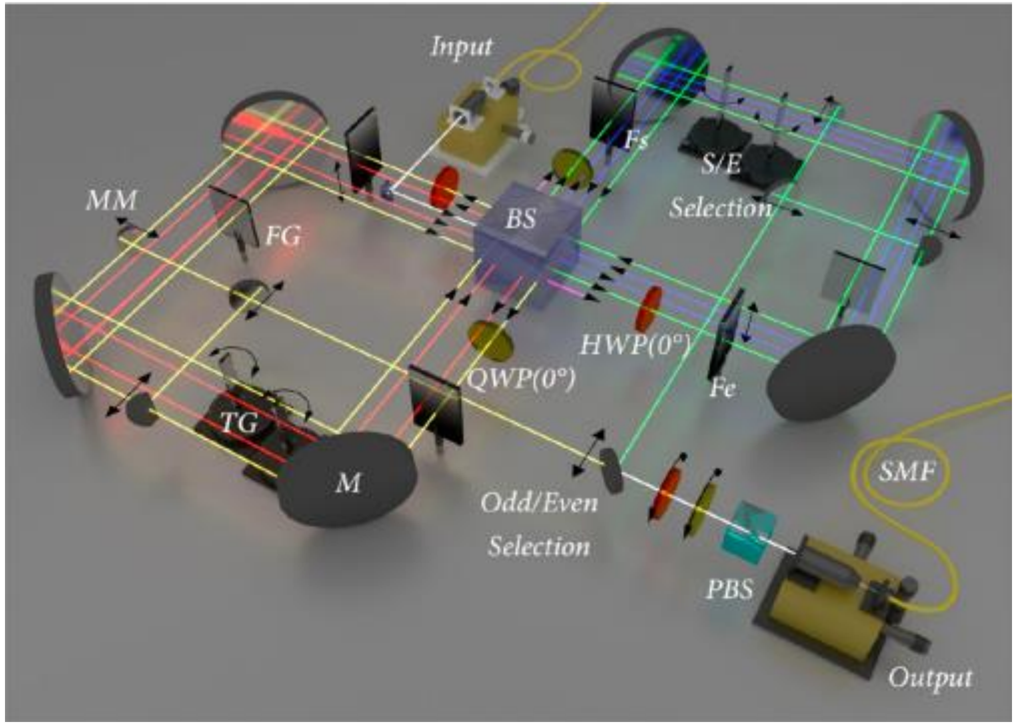
Álvaro Cuevas[1], Andrea Geraldi[1], Carlo Liorni[1,2], Luís Diego Bonavena[1], Antonella De Pasquale[3,4], Fabio Sciarrino[1], Vittorio Giovannetti[5] & Paolo Mataloni[1]





Input
Fs
S/E
Selection
MM
FG
BS
HWP(0°)
QWP(0°)
Fe
TG
M
SMF
Odd/Even
Selection
PBS
Output

PHYSICAL REVIEW VOLUME 129, NUMBER 4 15 FEBRUARY 1963

Relaxation Phenomena in Spin and Harmonic Oscillator Systems

JAYASEETHA RAU*†
Department of Physics, Brandeis University, Waltham, Massachusetts
(Received 2 August 1962)

A method is developed for generating relaxation by introducing a fundamental interval τ and a stirring hypothesis. The application to spin and harmonic oscillator systems is discussed in some detail. All the results are obtained by exact calculations without applying perturbation theory as the systems considered are simple and completely soluble. Equations similar to phenomenological Bloch equations are derived in the case of spin systems. The relaxation times obtained by the application of the theory are not only proportional to the strength of interaction, but also to the fundamental interval τ which plays an important role in the theory. It is shown that in the case of a harmonic oscillator system, an initial Boltzmann distribution relaxes to a final equilibrium Boltzman distribution through a sequence of transient Boltzman distributions.

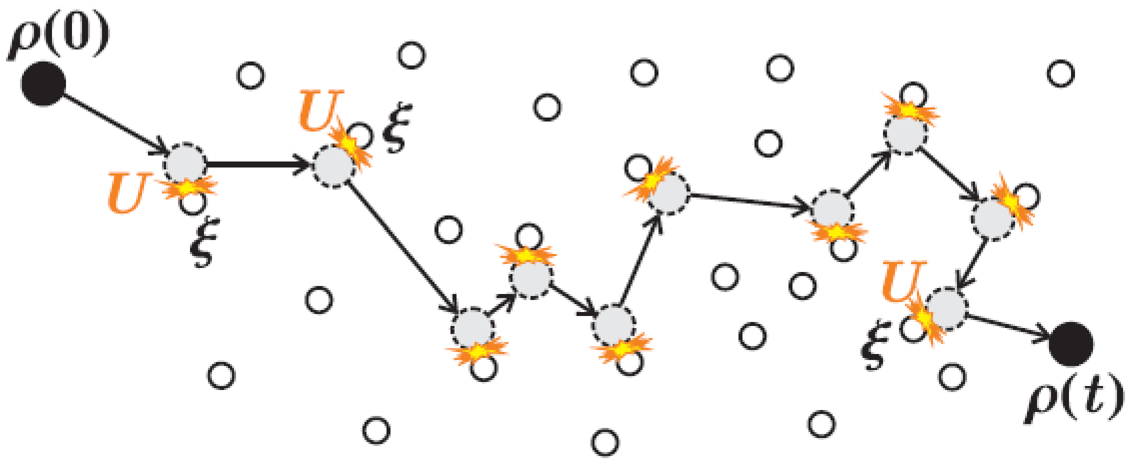


Figure from
SF, J. Piilo, S. Maniscalco, M. Ziman. Divisibility of quantum dynamical maps and collision models. Phys. Rev. A 96, 032111 (2017)

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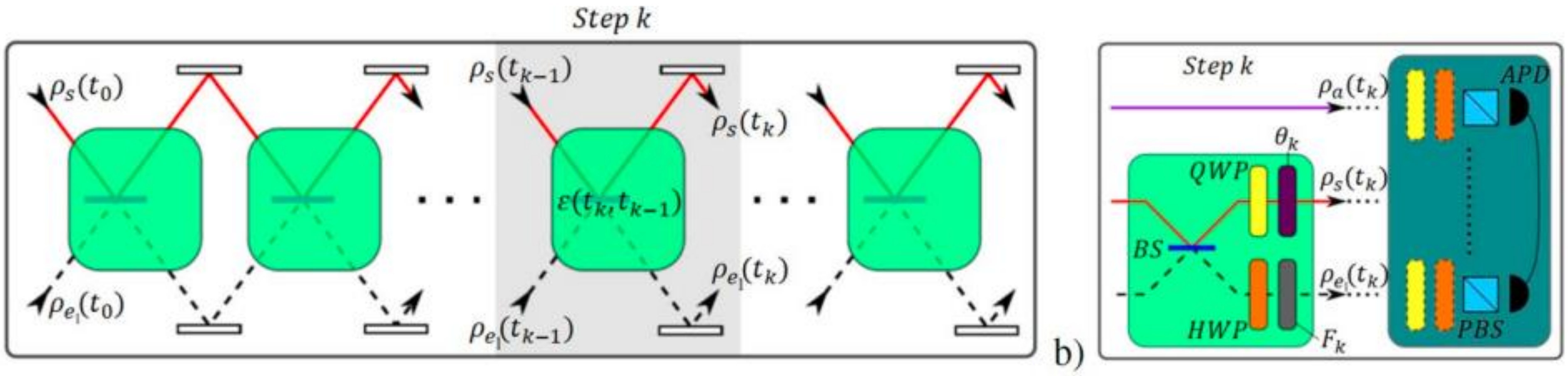
Francesco Ciccarello, Salvatore Lorenzo, Vittorio Giovannetti, G. Massimo Palma. Quantum collision models: open system dynamics from repeated interactions. arXiv:2106.11974 [quant-ph]

Steve Campbell, Bassano Vacchini. Collision models in open system dynamics: A versatile tool for deeper insights? EPL 133, 60001 (2021)

$$\widetilde{\varrho}(t) := \frac{\Lambda(t)[\varrho(0)]}{\mathrm{tr}\left[\Lambda(t)[\varrho(0)]\right]} \qquad \widetilde{\varrho}_{SA}(t) := \frac{\Lambda(t) \otimes \mathrm{Id}[|\psi_+\rangle\langle\psi_+|]}{\mathrm{tr}\left[\Lambda(t) \otimes \mathrm{Id}[|\psi_+\rangle\langle\psi_+|]\right]}$$



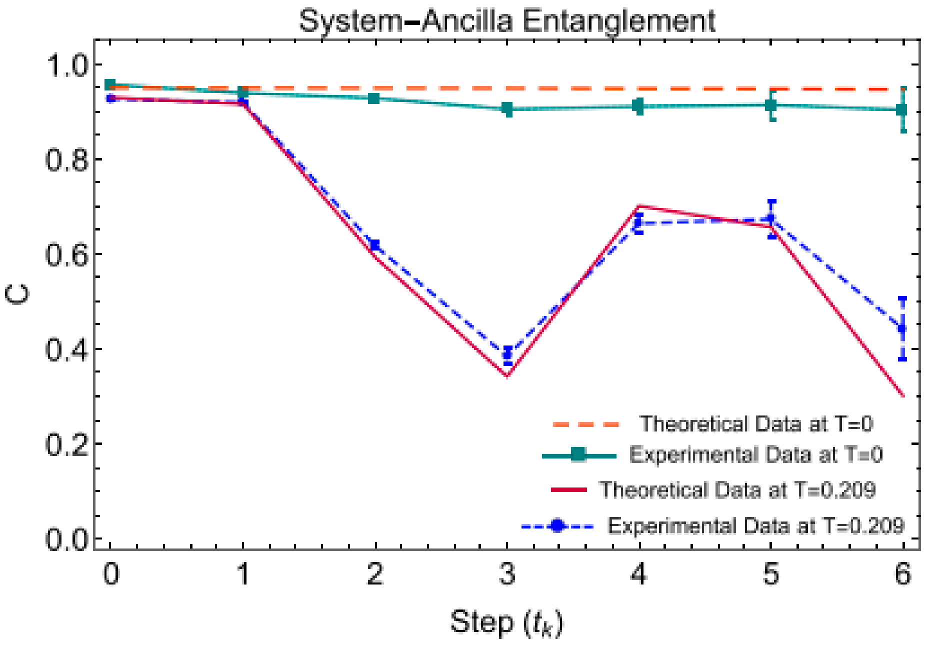


Figure 7. Evolution of Polarization Entanglement with reduced memory ($T=0$ and $T=0.209$). Concurrence of the single-photon sectors, post-select density matrix $\rho_{a,s}(t_k)$. All error bars were calculated from the propagation of 100 Monte-Carlo simulations with Poisson statistics, while theoretical data were simulated by considering the actual optical elements of the interferometric setup.

From the results of[14,36] we know in fact that in the cases where the relation $C_{a,s}(t_k) > C_{a,s}(t_{k-1})$ holds for some $k > 1$, a back-flow of information from e_1 to s has occurred, resulting in a clear indication of a non-Markovian character of the system dynamics. On the contrary a null increase of $C_{a,s}(t_k)$ cannot be used as an indication of Markovianity.

Rivas, A., Huelga, S. F. & Plenio, M. B. Entanglement and non-markovianity of quantum evolutions. Phys. Rev. Lett. 105, 050403 (2010).
Rivas, A., Huelga, S. F. & Plenio, M. B. Quantum non-markovianity: characterization, quantifcation and detection. Reports on Progress in Physics 77, 094001 (2014).
Lorenzo, S., Plastina, F. & Paternostro, M. Geometrical characterization of non-markovianity. Phys. Rev. A 88, 020102 (2013).

$$\frac{d\varrho(t)}{dt} = -\frac{1}{2}\left\{\gamma_H(t)\left|H\right\rangle\left\langle H\right| + \gamma_V(t)\left|V\right\rangle\left\langle V\right|, \varrho(t)\right\}$$

$$\Lambda(t)[\varrho(0)] \equiv \varrho(t) = \begin{pmatrix} e^{-\Gamma_H(t)}\varrho_{HH}(0) & e^{-\frac{1}{2}[\Gamma_H(t)+\Gamma_V(t)]}\varrho_{HV}(0) \\ e^{-\frac{1}{2}[\Gamma_H(t)+\Gamma_V(t)]}\varrho_{VH}(0) & e^{-\Gamma_V(t)}\varrho_{VV}(0) \end{pmatrix}$$

$$\Gamma_H(t) = \int_0^t \gamma_H(t')dt' \qquad \Gamma_V(t) = \int_0^t \gamma_V(t')dt'$$

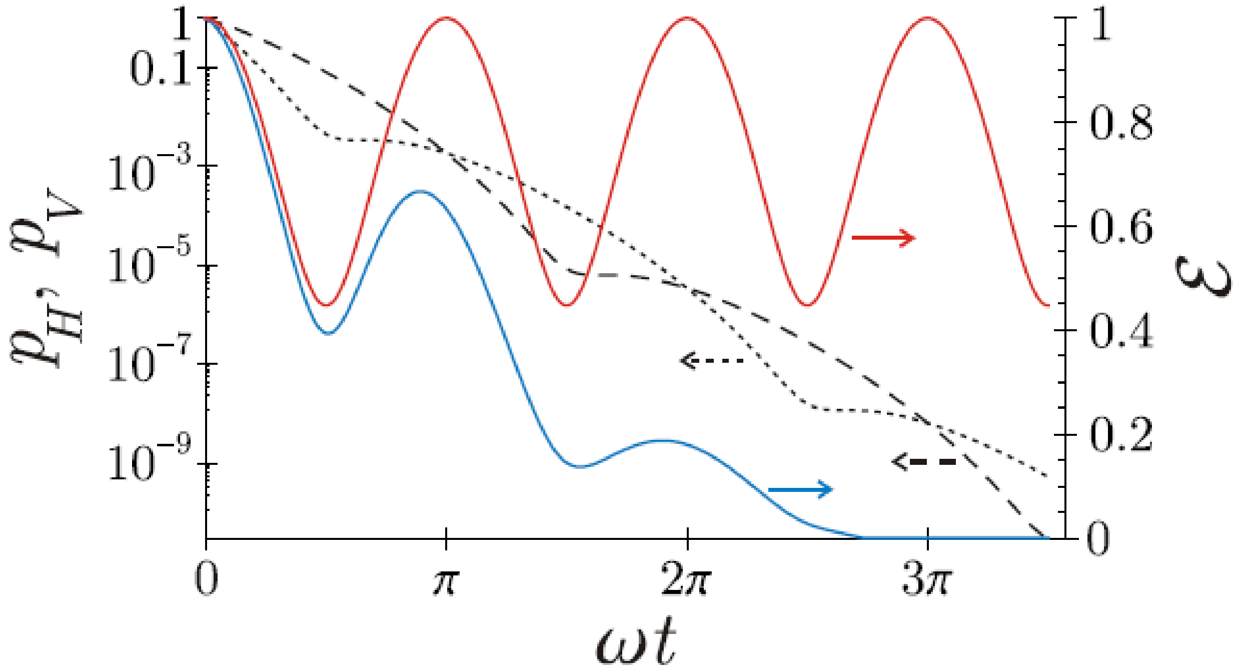
$$\gamma_H(t) = \gamma\left(1 - \frac{\omega\cos\omega t}{\sqrt{\gamma^2+\omega^2} + \gamma\sin\omega t}\right)$$

$$\gamma_V(t) = \gamma\left(1 + \frac{\omega\cos\omega t}{\sqrt{\gamma^2+\omega^2} - \gamma\sin\omega t}\right)$$

$$\gamma_H(t) \geq 0 \text{ and } \gamma_V(t) \geq 0 \text{ if } \gamma > 0,\ \omega > 0$$

$$p_H(t) := e^{-\Gamma_H(t)} = e^{-\gamma t}\left(1 + \frac{\gamma}{\sqrt{\gamma^2+\omega^2}}\sin\omega t\right)$$

$$p_V(t) := e^{-\Gamma_V(t)} = e^{-\gamma t}\left(1 - \frac{\gamma}{\sqrt{\gamma^2+\omega^2}}\sin\omega t\right)$$

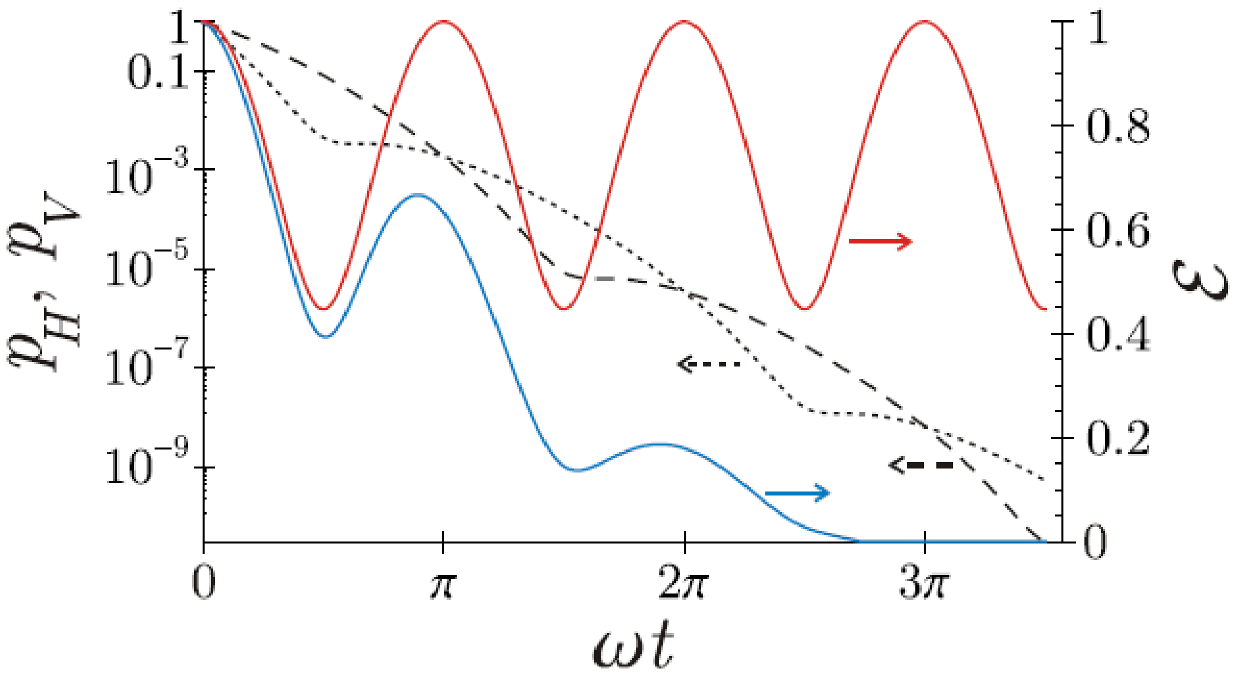


$$\mathcal{E}(t) = \frac{2\sqrt{p_H(t)p_V(t)}}{p_H(t)+p_V(t)} = \sqrt{\frac{\gamma^2\cos^2\omega t + \omega^2}{\gamma^2+\omega^2}}$$

$$\frac{\Lambda(t)\otimes \mathrm{Id}[|\psi_+\rangle\langle\psi_+|]}{\mathrm{tr}\left[\Lambda(t)\otimes \mathrm{Id}[|\psi_+\rangle\langle\psi_+|]\right]} = |\varphi(t)\rangle\langle\varphi(t)|,$$

$$|\varphi(t)\rangle = \frac{1}{\sqrt{p_H(t)+p_V(t)}}\left(\sqrt{p_H(t)}\,|H\rangle\otimes|H\rangle + \sqrt{p_V(t)}\,|V\rangle\otimes|V\rangle\right)$$

$$\frac{d\varrho(t)}{dt} = -\frac{1}{2}\left\{\gamma_H(t)\,|H\rangle\langle H| + \gamma_V(t)\,|V\rangle\langle V|\,,\varrho(t)\right\} + \frac{\lambda}{4}\sum_{i=1}^{3}\left(\sigma_i\varrho(t)\sigma_i - \varrho(t)\right)$$



Proposition 1. *Let $\Lambda(t)$ be a trace decreasing CP-divisible dynamical map and $\varrho_{SA}(0)$ be an initial system-ancilla state. If the postselected system-ancilla state $\Lambda(t_*) \otimes \mathrm{Id}[\varrho_{SA}(0)]/\mathrm{tr}\big[\Lambda(t_*) \otimes \mathrm{Id}[\varrho_{SA}(0)]\big]$ becomes separable at time moment t_*, then all future postselected states ($t \geq t_*$) remain separable.*

arXiv:2108.13372

Corollary 1. *Suppose the system experiences a trace decreasing dynamics $\Lambda(t)$. The system-ancilla entanglement death followed by the entanglement revival is an indication of CP-indivisibility for $\Lambda(t)$.*

arXiv:2108.13372

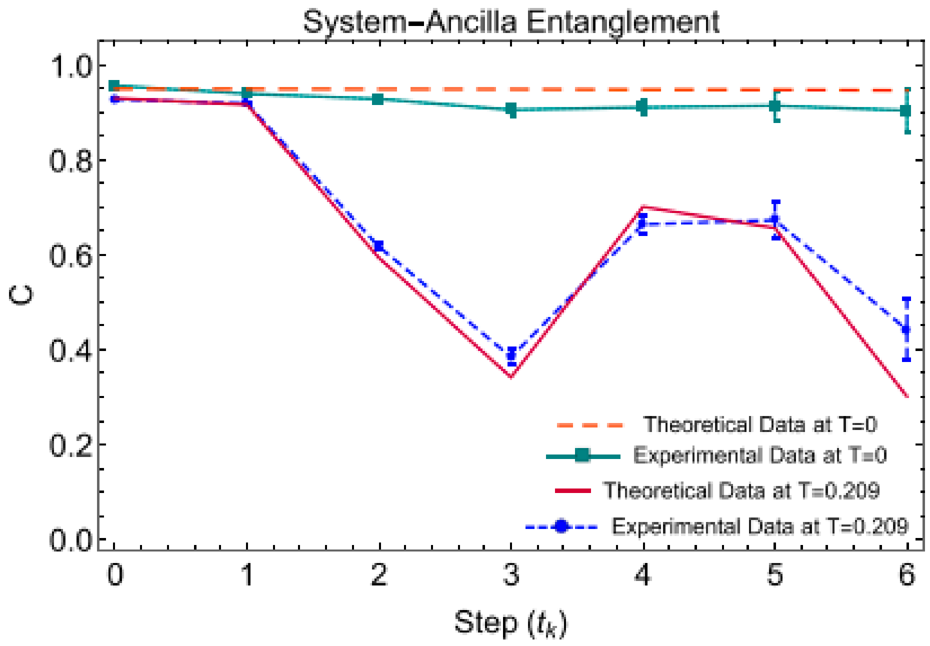


Figure 7. Evolution of Polarization Entanglement with reduced memory ($T=0$ and $T=0.209$). Concurrence of the single-photon sectors, post-select density matrix $\rho_{a,s}(t_k)$. All error bars were calculated from the propagation of 100 Monte-Carlo simulations with Poisson statistics, while theoretical data were simulated by considering the actual optical elements of the interferometric setup.

Generalizatoin of the backflow approach (BLP) to the trace-decreasing case

$$\Lambda_{\mathcal{D}}(t)[\varrho_1] = \Lambda(t)[\varrho_1]/\mathrm{tr}\left[\Lambda(t)[\varrho_1]\right]$$
$$\Lambda_{\mathcal{D}}(t)[\varrho_2] = \Lambda(t)[\varrho_2]/\mathrm{tr}\left[\Lambda(t)[\varrho_2]\right]$$

H.-P. Breuer, E.-M. Laine, J. Piilo, Measure for the degree of non-Markovian behavior of quantum processes in open systems, Phys. Rev. Lett. **103**, 210401 (2009)

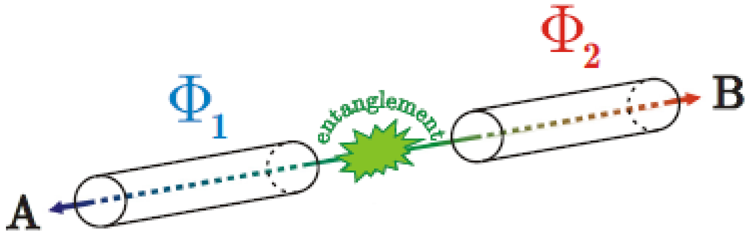
$$\frac{1}{2}\|\Lambda_{\mathcal{D}}(t)[\varrho_1] - \Lambda_{\mathcal{D}}(t)[\varrho_2]\|_1$$

$$\left\| \frac{p_1(t)}{p_1(t)+p_2(t)}\Lambda_{\mathcal{D}}(t)[\varrho_1] - \frac{p_2(t)}{p_1(t)+p_2(t)}\Lambda_{\mathcal{D}}(t)[\varrho_2] \right\|_1$$

$$p_i(t)\Lambda_{\mathcal{D}}(t)[\varrho_i] = \Lambda(t)[\varrho_i]$$

$$p_{\text{succ.dist.}} = \frac{1}{4}\left(\mathrm{tr}\left[\Lambda(t)[\varrho_1]\right] + \mathrm{tr}\left[\Lambda(t)[\varrho_2]\right] + \|\Lambda(t)[\varrho_1] - \Lambda(t)[\varrho_2]\|_1\right)$$

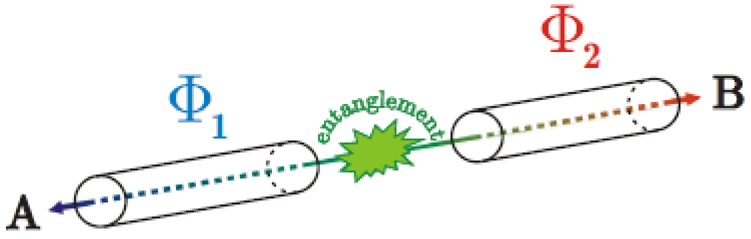
arXiv:2108.13372



SF, V. V. Frizen, D. V. Kolobova. Ultimate entanglement robustness of two-qubit states against general local noises. Phys. Rev. A 97, 012322 (2018)

$$\varrho_{12} = \sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \qquad \widetilde{\tau} = \max_{\varrho_{12}(0)} \tau$$

Problem to find the most robust entangled state.



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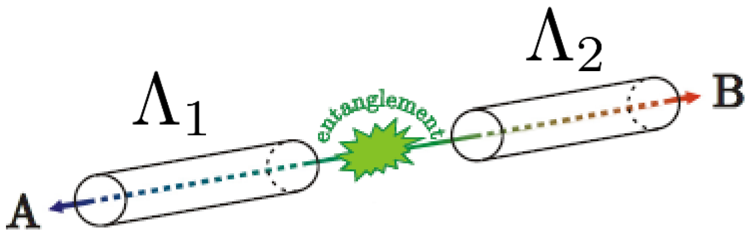
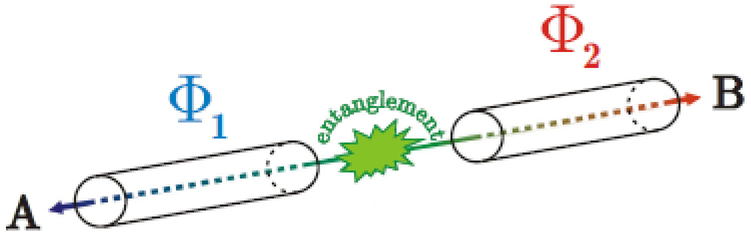


$$\frac{\Lambda_1(t) \otimes \Lambda_2(t)[\varrho_{12}(0)]}{\mathrm{tr}\,[\Lambda_1(t) \otimes \Lambda_2(t)[\varrho_{12}(0)]]}$$



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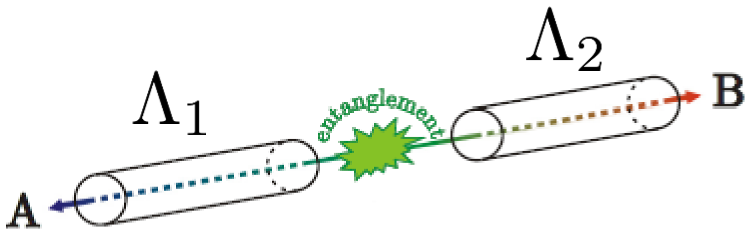


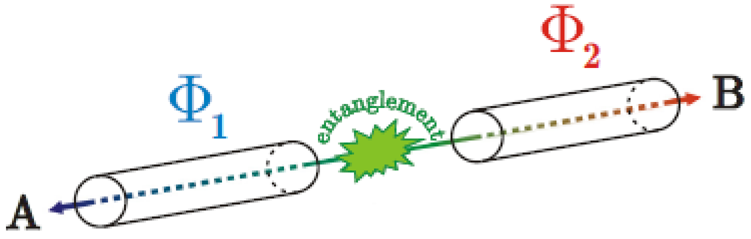
$$\frac{\Lambda_1(t) \otimes \Lambda_2(t)[\varrho_{12}(0)]}{\mathrm{tr}\,[\Lambda_1(t) \otimes \Lambda_2(t)[\varrho_{12}(0)]]}$$

Quantum Sinkhorn's theorem:

$$\Upsilon = \Phi_A \circ \Lambda \circ \Phi_B$$

$$\Phi_X[\varrho] \;=\; X \varrho X^\dagger$$

L. Gurvits, Classical complexity and quantum entanglement, J. Comput. Syst. Sci. 69, 448 (2004).
T. T. Georgiou and M. Pavon, Positive contraction mappings for classical and quantum Schrodinger systems, J. Math. Phys. 56, 033301 (2015).
G. Aubrun and S. J. Szarek, Two proofs of Stormer's theorem, arXiv:1512.03293 [math.FA] (2015).
G. Aubrun and S. J. Szarek, Alice and Bob Meet Banach: The Interface of Asympototic Geometry Analysis and Quantum Information Theory, section 2.4.3 (American Mathematical Society, 2017).



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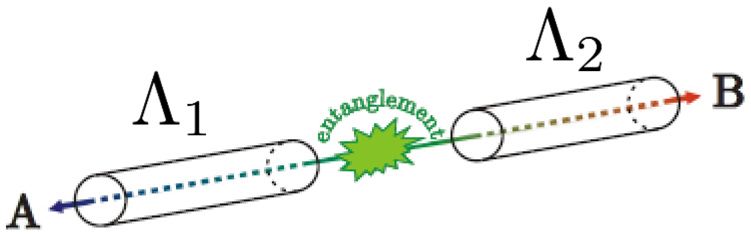

$$\frac{\Lambda_1(t) \otimes \Lambda_2(t)[\varrho_{12}(0)]}{\mathrm{tr}\,[\Lambda_1(t) \otimes \Lambda_2(t)[\varrho_{12}(0)]]}$$

Quantum Sinkhorn's theorem:

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$$\Upsilon \otimes \Upsilon' \qquad |\psi_{\Upsilon\otimes\Upsilon'}\rangle = \frac{1}{\sqrt{2}}(|\varphi\rangle \otimes |\chi\rangle + |\varphi_\perp\rangle \otimes |\chi_\perp\rangle)$$

$$|\psi_{\Lambda\otimes\Lambda'}\rangle = \frac{B(\widetilde{\tau}) \otimes B'(\widetilde{\tau})|\psi_{\Upsilon\otimes\Upsilon'}\rangle}{\sqrt{\langle\psi_{\Upsilon\otimes\Upsilon'}|B^\dagger(\widetilde{\tau})B(\widetilde{\tau}) \otimes B'(\widetilde{\tau})^\dagger B'(\widetilde{\tau})|\psi_{\Upsilon\otimes\Upsilon'}\rangle}}$$

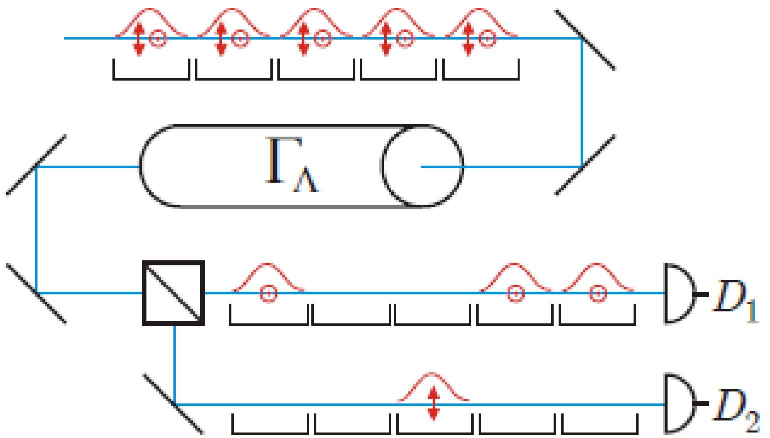
SF. Entanglement robustness in trace decreasing quantum dynamics. Quanta 10, 15-21 (2021)

Quantum communication rate problem

Polarization dependent losses:

$$\Lambda[\varrho] = A\varrho A^\dagger, \quad A = \sqrt{p_H}|H\rangle\langle H| + \sqrt{p_V}|V\rangle\langle V|$$

$$\Gamma_\Lambda[\rho] = \Lambda[\rho] \oplus \mathrm{tr}\left[\rho - \Lambda[\rho]\right]|e\rangle\langle e| = \begin{pmatrix} \Lambda[\rho] & \mathbf{0} \\ \mathbf{0}^\top & \mathrm{tr}\left[\rho(I - \Lambda^\dagger[I])\right] \end{pmatrix}$$



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If $\Lambda = p\mathrm{Id}$, $0 \leq p \leq 1$, then $\Gamma_{p\mathrm{Id}}$ is nothing else but the conventional erasure channel [M. Grassl, T. Beth, and T. Pellizzari, Codes for the quantum erasure channel, Phys. Rev. A 56, 33 (1997)]; [C. H. Bennett, D. P. DiVincenzo, and J. A. Smolin, Capacities of quantum erasure channels, Phys. Rev. Lett. 78, 3217 (1997)].

If $\Lambda = p\Phi$, where $\Phi : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_2)$ is a dephasure channel, then $\Gamma_{p\Phi}$ is a so-called dephrasure channel [F. Leditzky, D. Leung, and G. Smith, Dephrasure channel and superadditivity of coherent information, Phys. Rev. Lett. 121, 160501 (2018)].

$\Lambda = p\Phi$, where $\Phi : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_2)$ is a general channel or an amplitude damping channel, in particular [V. Siddhu and R. B. Griffiths, Positivity and nonadditivity of quantum capacities using generalized erasure channels, arXiv:2003.00583].

In contrast to these specific cases, Λ **does not have to be unbiased**, so the erasure probability $\mathrm{tr}\left[\varrho(I - \Lambda^\dagger[I])\right]$ is **state-dependent** [S.N. Filippov. J. Phys. A: Math. Theor. 54, 255301 (2021)]

$$\Psi[\varrho] = \mathrm{tr}_E\big[V\varrho V^\dagger\big], \qquad \widetilde{\Psi}[\varrho] = \mathrm{tr}_B\big[V\varrho V^\dagger\big]$$

$$Q(\Phi) = \lim_{n\to\infty} \frac{1}{n} Q_1(\Phi^{\otimes n}), \quad Q_1(\Psi) = \sup_{\rho}\{S(\Psi[\rho]) - S(\widetilde{\Psi}[\rho])\}$$

The quantum capacity is known to satisfy the additivity property $Q(\Phi) = Q_1(\Phi)$ if Φ is gedradable, i.e., there exists a quantum channel Ξ such that $\widetilde{\Phi} = \Xi \circ \Phi$ [I. Devetak, P. W. Shor. The capacity of a quantum channel for simultaneous transmission of classical and quantum information. Commun. Math. Phys. 256, 287 (2005)]

If Φ is antigedradable, i.e., there exists a quantum channel Ξ such that $\Phi = \Xi \circ \widetilde{\Phi}$, then $Q(\Phi) = 0$ and the additivity property is trivially fulfilled.

$$Q(\Phi) = \lim_{n\to\infty} \frac{1}{n} Q_1(\Phi^{\otimes n}), \quad Q_1(\Psi) = \sup_{\rho \in \mathcal{D}(\mathcal{H}_{d'})} \{S(\Psi[\rho]) - S(\widetilde{\Psi}[\rho])\}$$

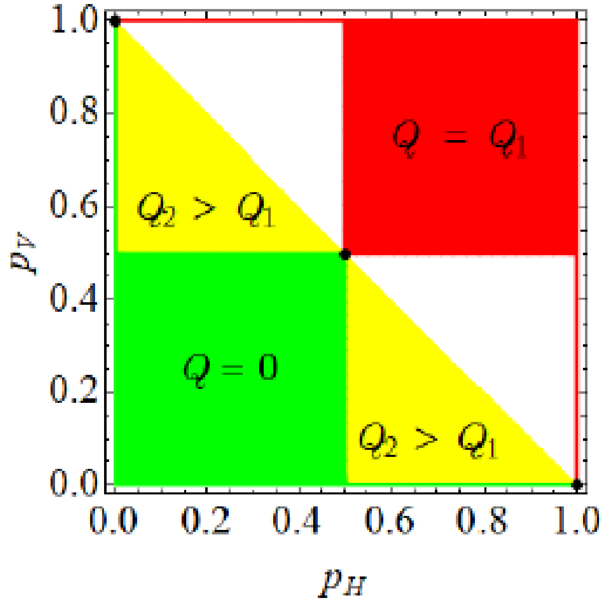
The **superadditivity** of coherent information, i.e., the strict inequality $Q_1(\Phi^{\otimes n}) > nQ_1(\Phi)$, is known to hold for

- some depolarizing channels if $n \geq 3$
- some dephrasure channels if $n \geq 2$ $(\frac{1}{2}Q_1(\Phi^{\otimes 2}) - Q_1(\Phi) \approx 2.5 \cdot 10^{-3})$
- concatenations of the erasure channel with the amplitude damping channels $(\frac{1}{2}Q_1(\Phi^{\otimes 2}) - Q_1(\Phi) \approx 5 \cdot 10^{-3})$
- the state-of-the-art channels $\Phi : \mathcal{B}(\mathcal{H}_3) \to \mathcal{B}(\mathcal{H}_3)$ with $\frac{1}{2}Q_1(\Phi^{\otimes 2}) - Q_1(\Phi) \approx 4.4 \cdot 10^{-2}$ and their higher-dimensional generalizations
- collection of peculiar channels if $n \geq n_0$, where $n_0 \geq 2$ specifies the channel and can be arbitrary

$$\Lambda[\varrho] = A\varrho A^{\dagger}, \quad A = \sqrt{p_H}|H\rangle\langle H| + \sqrt{p_V}|V\rangle\langle V|$$

$$\Gamma_{\Lambda}[\rho] = \begin{pmatrix} A\rho A^{\dagger} & \mathbf{0} \\ \mathbf{0}^{\top} & \mathrm{tr}\left[\sqrt{I - A^{\dagger}A}\,\rho\,\sqrt{I - A^{\dagger}A}\right] \end{pmatrix}$$

$$\widetilde{\Gamma}_{\Lambda}[\rho] = \begin{pmatrix} \sqrt{I - A^{\dagger}A}\,\rho\,\sqrt{I - A^{\dagger}A} & \mathbf{0} \\ \mathbf{0}^{\top} & \mathrm{tr}\left[A\rho A^{\dagger}\right] \end{pmatrix}$$



S. N. Filippov. Capacity of trace decreasing quantum operations and superadditivity of coherent information for a generalized erasure channel. J. Phys. A: Math. Theor. 54, 255301 (2021)

Proposition

Let $\Lambda \in \mathcal{O}(\mathcal{H}_2)$ be a quantum operation describing polarization dependent losses with parameters p_H and p_V. The strict inequality $\frac{1}{2}Q_1(\Gamma_\Lambda^{\otimes 2}) > Q_1(\Gamma_\Lambda)$ holds if either $\frac{1}{2} < p_H < 1$ and $0 < p_V < 1 - p_H$ or $\frac{1}{2} < p_V < 1$ and $0 < p_H < 1 - p_V$.

Proof.

$$\varrho_{\rm opt}^{(1)} = \varrho_{HH}|H\rangle\langle H| + \varrho_{VV}|V\rangle\langle V|$$

$$\begin{aligned}
\varrho^{(2)} &= \left(\varrho_{\rm opt}^{(1)}\right)^{\otimes 2} + \varrho_{HH}\varrho_{VV}\left(|HV\rangle\langle VH| + |VH\rangle\langle HV|\right) \\
&= \varrho_{HH}^2|HH\rangle\langle HH| + \varrho_{VV}^2|VV\rangle\langle VV| + 2\varrho_{HH}\varrho_{VV}\frac{|HV\rangle + |VH\rangle}{\sqrt{2}}\frac{\langle HV| + \langle VH|}{\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
S(\Gamma^{\otimes 2}[\varrho^{(2)}]) &= S\left((\Gamma[\varrho_{\rm opt}^{(1)}])^{\otimes 2}\right) - (2\log 2)p_H p_V \varrho_{HH}\varrho_{VV} \\
S(\widetilde{\Gamma}^{\otimes 2}[\varrho^{(2)}]) &= S\left((\widetilde{\Gamma}[\varrho_{\rm opt}^{(1)}])^{\otimes 2}\right) - (2\log 2)(1-p_H)(1-p_V)\varrho_{HH}\varrho_{VV}
\end{aligned}$$

Suppose $p_H > p_V$.

Consider the subspace $\mathcal{H}_{n-1,1}$ spanned by the vector $|H\rangle^{\otimes(n-1)} \otimes |V\rangle$ and all its permutations.

$$|W^{(n)}\rangle = \frac{1}{\sqrt{n}}\left(|HH\ldots HHV\rangle + |HH\ldots HVH\rangle + \ldots + |VH\ldots HHH\rangle\right) \in \mathcal{H}_{n-1,1}.$$

$$\varrho^{(n)}|\varphi\rangle = \begin{cases} (\varrho_{\mathrm{opt}}^{(1)})^{\otimes n}|\varphi\rangle & \text{if } \langle\chi|\varphi\rangle = 0 \text{ for all } |\chi\rangle \in \mathcal{H}_{n-1,1}, \\ \varrho_{HH}^{n-1}\varrho_{VV} n\langle W^{(n)}|\varphi\rangle\, |W^{(n)}\rangle & \text{if } |\varphi\rangle \in \mathcal{H}_{n-1,1} \end{cases}$$

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$$\begin{aligned} \Gamma^{\otimes n} &= \Lambda_F^{\otimes n} \oplus \ldots \\ & \underbrace{\oplus \left[\Lambda_F^{\otimes(n-k)} \otimes (\mathrm{Tr} \circ \Lambda_G)^{\otimes k} \right] \oplus \ldots}_{\binom{n}{k} \text{ terms}} \\ & \oplus \ldots \oplus (\mathrm{Tr} \circ \Lambda_G)^{\otimes n} \end{aligned}$$

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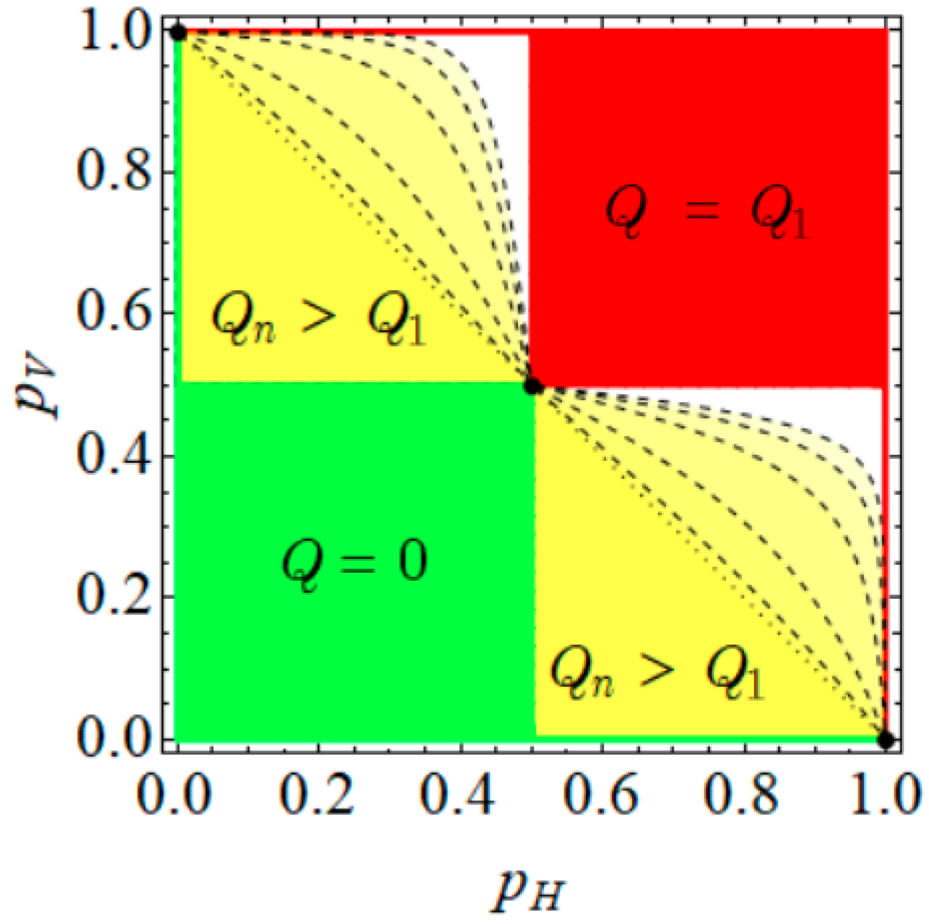
$$Q_n(\Gamma) - Q_1(\Gamma) \geq \varrho_{HH}^{n-1}\varrho_{VV}\sum_{k=0}^{n-1}\binom{n-1}{k}(1-p_H)^{n-k-1}p_H^k \times [(1-p_V)\log(n-k) - p_V\log(k+1)]$$



$$\frac{1}{n}I_c(\varrho^{(n)},\Gamma^{\otimes n}) \approx Q_1(\Gamma) + \begin{cases} (1-2p_V)\varrho_{HH}^{n-1}\varrho_{VV}\log n & \text{if } p_H > p_V, \\ (1-2p_H)\varrho_{HH}\varrho_{VV}^{n-1}\log n & \text{if } p_V > p_H. \end{cases}$$

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Summary and conclusions

- Trace decreasing quantum operations are physical.
- Experiments involving postselection should be revisited.
- Concept of the generalized erasure channel is introduced.
- Analytical proof for the superadditivity of coherent information, i.e., a strict separation between the single-letter quantum capacity and an *n*-letter quantum capacity.

Thank you for listening!