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Part II: Based on r-function approach

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Gen Kimura (Shibaura Institute of Technology)

Torun, 2016

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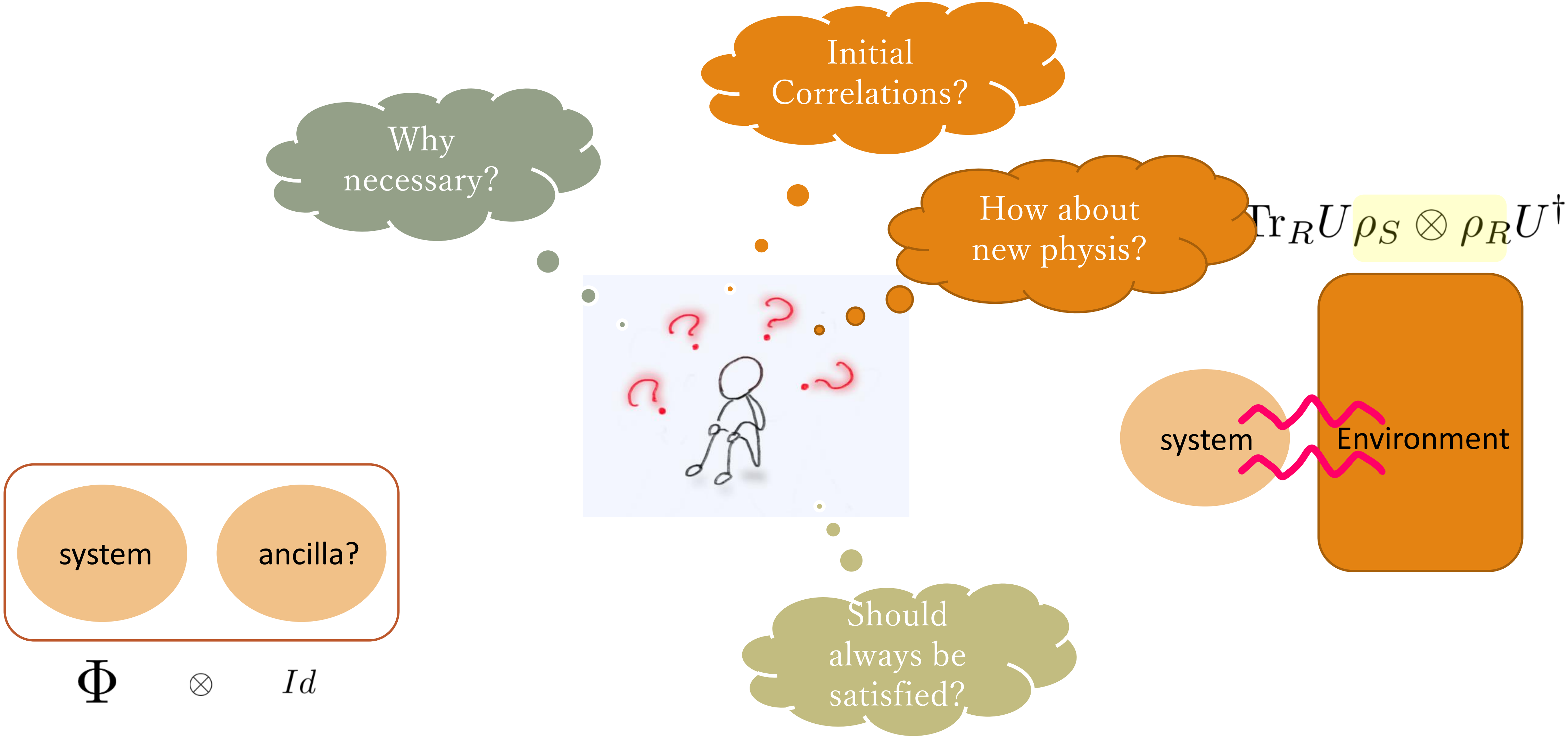
Why we need completely positivity condition?


Why necessary?
Initial Correlations?
How about new physis?
Should always be satisfied?
$\mathrm{Tr}_R U \rho_S \otimes \rho_R U^\dagger$
system
Environment
system
ancilla?
$\Phi \otimes Id$

George would never take a postulate for granted.

quoted by Prof. Pascazios slide yesterday

According to the discipline of natural science,
the validity of a theory is ultimately determined by experiment.

Physics of CP condition !

According to the discipline of natural science,
the validity of a theory is ultimately determined by experiment.

CP

Markovianity

Goal

Experimentally testable condition.

$$\Gamma_\alpha \leq \heartsuit \sum_{\beta=1}^{d^2-1} \Gamma_\beta$$

Γ_α: Relaxation Rate

⇔ Quantum Dynamical Semigroup

Bell Theorem

Reality

Locality

"Free will"

Bell's Inequality

$$\langle AC \rangle + \langle AD \rangle + \langle BC \rangle - \langle BD \rangle \leq 2$$

$\langle AC \rangle, \cdots$ Correlation

Quantum dynamical semigroup.. General Markovian CP quantum dynamics

1) Completely Positive Trace Preserving Map $\rho \mapsto \rho_t = \Lambda_t \rho$

2) One parameter (time) Dynamical Semigroup $\Lambda_{t+s} = \Lambda_t \Lambda_s \ (\forall t, s \geq 0)$

Hille-Yoshida (1948)

➡ $$\frac{d\rho}{dt} = \mathcal{L}\rho \quad \text{s.t.} \quad \Lambda_t = \exp(t\mathcal{L})$$

[Thm] (GKLS 1976) Generator of quantum dynamical semigroup is always written

$$\mathcal{L} = \mathcal{H} + \mathcal{D}$$

* Hamiltonian Part

$$\mathcal{H}(\rho) = -i[H, \rho] \quad \text{where} \quad H = H^\dagger$$

(effective) Hamiltonian

* Dissipative Part: $\tilde{L}_k := \sqrt{\gamma_k} L_k$

$$\mathcal{D}(\rho) = \frac{1}{2}\sum_k \gamma_k (2L_k\rho L_k^\dagger - L_k^\dagger L_k \rho - \rho L_k^\dagger L_k) \quad \text{where} \quad \gamma_k \geq 0$$

L_k: Jump/Noise Operator

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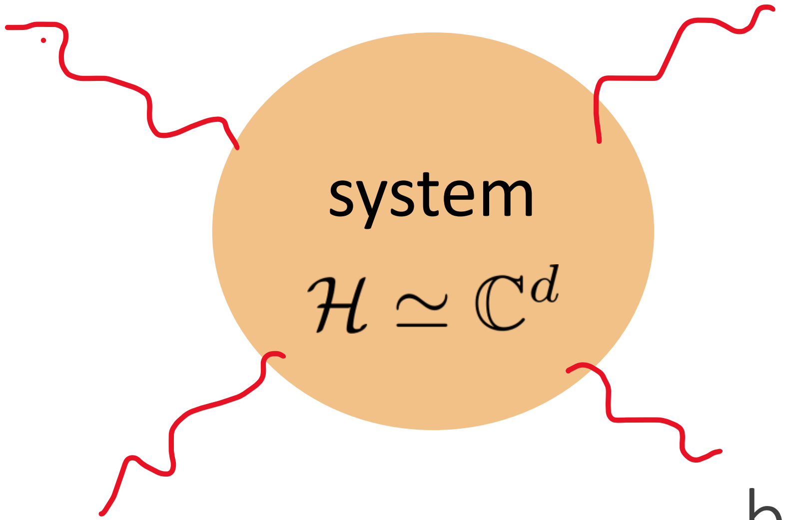
$$\mathcal{D}(\rho) = \frac{1}{2}\sum_k (2L_k\rho L_k^\dagger - L_k^\dagger L_k \rho - \rho L_k^\dagger L_k) \qquad \mathrm{tr}\ L_k = 0$$

L_k: Jump/Noise Operator

In this work, we restrict ourself to a d-level quantum system

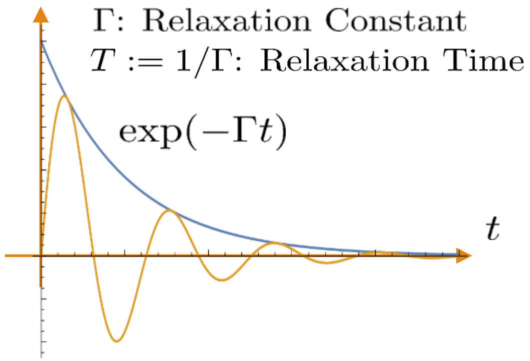


but for arbitrary finite d !!

Physics of GKLS Master equation

We focus on Relaxation Times !

Time Evolution of any physical quantities = “super position” of



Decaying time determined by Eigenvalues of Generator: $\mathcal{L}u_\alpha = \lambda_\alpha u_\alpha \ (u_\alpha \neq 0)$

$$\lambda_0 = 0 \quad \& \quad \lambda_\alpha = -\Gamma_\alpha + i\omega_\alpha \quad (\alpha = 1, \ldots, d^2 - 1)$$

$\Gamma_\alpha := -\mathrm{Re}\lambda_\alpha$ Relaxation Rates

$T_\alpha := 1/\Gamma_\alpha$ Relaxation Times

Main Results: r- function approach

[Theorems] For any d-level GKLS master equation,

$$\Gamma_\alpha \leq \frac{1+\sqrt{2(1-\frac{1}{d})}}{2d}\sum_{\beta=1}^{d^2-1}\Gamma_\beta \leq \frac{1+\sqrt{2}}{2d}\sum_{\beta=1}^{d^2-1}\Gamma_\beta \leq \frac{\sqrt{2}}{d}\sum_{\beta=1}^{d^2-1}\Gamma_\beta$$

Neceesary condition for CP condition and Markovianity testable by experiments

Kimura, Ajisaka, Watanabe (2017) Chruscinski, Fujii, Kimura, Ohno (2021) r stand for "relaxation"

[Definition] (r-function) For complex matrices $A, B \in M_d(\mathbb{C})$, we define

$$r(A,B) := \frac{1}{2}\operatorname{tr}(A^\dagger A B^\dagger B + A A^\dagger B^\dagger B - A^\dagger B A B^\dagger - B A^\dagger B^\dagger A)$$

$$\begin{aligned} r(A,B) &= \frac{1}{2}\operatorname{tr}(\{A, A^\dagger\}B^\dagger B) - \Re\operatorname{tr}(A^\dagger B A B^\dagger), \\ &= \frac{1}{2}(\langle [B,A] | BA\rangle + \langle [B, A^\dagger] | BA^\dagger\rangle), \\ &= \frac{1}{2}(\|[A,B]\|^2 + \operatorname{tr} A^\dagger A [B^\dagger, B]) \end{aligned}$$

Commutator

$$[A,B] := AB - BA$$

Anti-commutator

$$\{A,B\} := AB + BA$$

Hilbert-Schmidt Inner Prod.

$$\langle A, B\rangle := \operatorname{tr} A^\dagger B$$

Frobenius (Hilbert-Schmidt) Norm

$$\|A\| := \sqrt{\operatorname{tr} A^\dagger A}$$

Properties of r-function

- Unitry Invariance: $r(UAU^\dagger, UBU^\dagger) = r(A, B)$
- $r(\alpha A, \beta B) = |\alpha|^2|\beta|^2 r(A, B)$
- For Cartesian decomposition $A = A_R + iA_I$,

$$r(A, B) = r(A_R, B) + ir(A_I, B)$$

[Lemma 1] $$\sum_{\alpha=1}^{d^2-1} \Gamma_\alpha = d \sum_k \|L_k\|^2$$

Frobenius Norm:

$$\|A\| := \sqrt{\mathrm{tr}\, A^\dagger A}$$

Complex Eigenvalues appears as conjugate pair & leading eigenvalue 0

- ♥ $$\sum_{\alpha=1}^{d^2-1} \Gamma_\alpha = -\sum_{\alpha=0}^{d^2-1} \lambda_\alpha = -\mathrm{tr}\mathcal{L} \qquad \mathcal{L} u_\alpha = \lambda_\alpha u_\alpha \ (u_\alpha \neq 0)$$

- ♥ $$\mathrm{tr}\mathcal{L} = -d \sum_k \|L_k\|^2$$ Wolf and Cirac (2008)

Tensor rep. (c.f. Prof. Lakshminarayan talk yesterday) and Tr L_k = 0

$$\mathcal{L}(\rho) = -i[H,\rho] + \frac{1}{2}\sum_k (2L_k \rho L_k^\dagger - L_k^\dagger L_k \rho - \rho L_k^\dagger L_k)$$

$$\mapsto \hat{\mathcal{L}} = -i(I \otimes H - H^T \otimes I) + \frac{1}{2}\sum_k (2\overline{L_k} \otimes L_k - \mathbb{I} \otimes L_k^\dagger L_k - L_k^T \overline{L_k} \otimes \mathbb{I})$$

[Lemma 2] $\Gamma_\alpha = \frac{1}{\|u_\alpha\|^2} \sum_k r(u_\alpha, L_k)$

$\mathcal{L} u_\alpha = \lambda_\alpha u_\alpha \ (u_\alpha \neq 0)$

Reason why we call r-function!

- Re $\mathrm{tr}\left(u_\alpha^\dagger \times \quad \lambda_\alpha u_\alpha = \mathcal{L}(u_\alpha) = -i[H, u_\alpha] + \frac{1}{2}\sum_k (2L_k u_\alpha L_k^\dagger - L_k^\dagger L_k u_\alpha - u_\alpha L_k^\dagger L_k) \right)$

$$\Rightarrow \Gamma_\alpha = \frac{1}{2\|u_\alpha\|^2} \sum_k \mathrm{tr}(u_\alpha^\dagger u_\alpha L_k^\dagger L_k + u_\alpha u_\alpha^\dagger L_k^\dagger L_k - u_\alpha^\dagger L_k u_\alpha L_k^\dagger - L_k u_\alpha^\dagger L_k^\dagger u_\alpha)$$

$$= \frac{1}{\|u_\alpha\|^2} \sum_k r(u_\alpha, L_k)$$

$\Gamma_\alpha := -\mathrm{Re}\lambda_\alpha \ \& \ \|u_\alpha\|^2 = \mathrm{tr} u_\alpha^\dagger u_\alpha$

[Proposition 3] $r(A, B) \leq c(d)\|A\|^2\|B\|^2$

$\Rightarrow$ For any GKLS master eq., $\Gamma_\alpha \leq \frac{c(d)}{d}\sum_\beta \Gamma_\beta$

[Lemma 2] $$\Gamma_\alpha = \frac{1}{\|u_\alpha\|^2}\sum_k r(u_\alpha, L_k)$$

$$\leq c(d)\sum_k \|L_k\|^2$$

$$= \frac{c(d)}{d}\sum_\beta \Gamma_\beta$$

←[Lemma 1] $\displaystyle\sum_{\beta=1}^{d^2-1}\Gamma_\beta = d\sum_k \|L_k\|^2$

[Prop. 4] For any matrices A, B,

$$r(A,B) \leq \sqrt{2}||A||^2||B||^2$$

$c(d)$

Proof.

$$r(A,B) = \frac{1}{2}(\langle [B,A]|BA\rangle + \langle [B,A^\dagger]|BA^\dagger\rangle)$$

$$\leq \frac{1}{2}(\|[B,A]\|\|B\|\|A\| + \|[B,A^\dagger]\|\|B\|\|A^\dagger\|$$

(using triangle, Schwarz, and norm inequalities.,)

$$\leq \sqrt{2}\|A\|^2\|B\|^2$$

$\|[A,B]\|^2 \leq 2\|A\|^2\|B\|^2$

(Böttcher-Wenzel Inequality)

[Prop. 3] $r(A,B) \leq c(d)\|A\|^2\|B\|^2$

$$\Rightarrow \Gamma_\alpha \leq \frac{c(d)}{d}\sum_\beta \Gamma_\beta$$

[Theorem 5] For any d-level GKLS,

$$\Gamma_\alpha \leq \frac{\sqrt{2}}{d}\sum_{\beta=1}^{d^2-1}\Gamma_\beta$$

With r-function approach

$$r(A, B) \leq c(d) \|A\|^2 \|B\|^2$$

Task: Find the best (minimum) constant c(d),
such that inequality is saturated by some A and B

$$c_{\mathrm{opt}}(d) = \sup_{A,B} \frac{r(A, B)}{\|A\|^2 \|B\|^2}$$

[Prop. 6] For any A, B,

$$r(A,B) \leq \frac{1+\sqrt{2}}{2}\|A\|^2\|B\|^2$$

The inequality is achieved by some A and B.

By unitary invariance and Cartesian decomposition, one can restrict matrix A to be diagonal:

$$A = \mathrm{diag}[a_1, a_2, ..., a_n] \quad B = \{b_{ij}\}_{i,j=1}^n$$

$$r(A,B) \leq k||A||^2||B||^2 \Rightarrow k(\sum_k a_k^2)(\sum_{i=j}|b_{ji}|^2) - \sum_{i\neq j=1}|b_{ji}|^2(a_i - a_i a_j)$$

$$\boxed{k = \frac{1+\sqrt{2}}{2}}$$

$$\geq \sum_{i\neq j=1}|b_{ji}|^2(\sqrt{k-1}a_i - \sqrt{k}a_j)^2 \geq 0$$

Achievability is also easity shown.

[Prop. 6] For any A, B,

$$r(A,B) \leq \frac{1+\sqrt{2}}{2}\|A\|^2\|B\|^2$$

The inequality is achieved by some A and B.

[Prop. 3] $r(A,B) \leq c(d)\|A\|^2\|B\|^2$

$$\Rightarrow \Gamma_\alpha \leq \frac{c(d)}{d}\sum_\beta \Gamma_\beta$$

[Theorem 7] For any d-level GKLS,

$$\Gamma_\alpha \leq \frac{1+\sqrt{2}}{2d}\sum_{\beta=1}^{d^2-1}\Gamma_\beta$$

One may anticipate that this bound can no more be improved with r-function approach.

[Prop. 3] $r(A,B) \le c(d)\|A\|^2\|B\|^2 \quad \Rightarrow \Gamma_\alpha \le \frac{c(d)}{d}\sum_\beta \Gamma_\beta$

$r(u_\alpha, L_k)$

A can be restricted to traceless !!

Eigenvector belonging to non-zero eigenvalue is traceless

$$\lambda_\alpha \neq 0 \Rightarrow \mathrm{tr}\ u_\alpha = 0$$

Trace Preserving Property

[Prop. 8] For any complex matrices $A, B \in M_d(\mathbb{C})$ with $\mathrm{tr}A = 0$,

$$r(A,B) \leq \frac{1+\sqrt{2(1-\frac{1}{d})}}{2}\|A\|^2\|B\|^2$$

where the equality can be achieved by some A and B.

[Prop. 3] $r(A,B) \leq c(d)\|A\|^2\|B\|^2$

$$\Rightarrow \Gamma_\alpha \leq \frac{c(d)}{d}\sum_\beta \Gamma_\beta$$

$$d = 2 \Rightarrow \Gamma_\alpha \leq \frac{1}{2}\sum_{\beta=1}^{d^2-1}\Gamma_\beta$$

Kimura (2002)

[Theorem 9] For any d-level GKLS,

$$\Gamma_\alpha \leq \frac{1+\sqrt{2(1-\frac{1}{d})}}{2}\sum_{\beta=1}^{d^2-1}\Gamma_\beta$$

Conclusion

Based on r-function approach, we have found universal constraints for relaxation rates

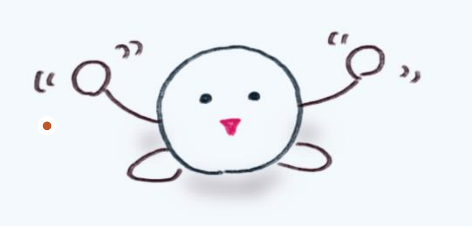
Necessary Condition for CP condition and Markovianity testable by experiments

$$\Gamma_\alpha \le \frac{1}{d}\sum_{\beta=1}^{d^2-1}\Gamma_\beta \le \frac{1+\sqrt{2(1-\frac{1}{d})}}{2d}\sum_{\beta=1}^{d^2-1}\Gamma_\beta \le \frac{1+\sqrt{2}}{2d}\sum_{\beta=1}^{d^2-1}\Gamma_\beta \le \frac{\sqrt{2}}{d}\sum_{\beta=1}^{d^2-1}\Gamma_\beta$$

Tightest Conjecture ??

Still Conjecture !!

Thank you for your kind attention !!



Universal Constraints for GKLS generator $c(d)\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta$