On the universal constraints for relaxation rates for quantum dynamical semigroup

Part I: A physical manifestation of complete positivity

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Stochastic Dynamics of Quantum Mechanical Systems

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Stochastic Dynamics of Quantum-Mechanical Systems

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(Received August 15, 1960)

The most general dynamical law for a quantum mechanical system with a finite number of levels is formulated. A fundamental role is played by the so-called "dynamical matrix" whose properties are stated in a sequence of theorems. A necessary and sufficient criterion for distinguishing dynamical matrices corresponding to a Hamiltonian time-dependence is formulated. The non-Hamiltonian case is discussed in detail and the application to paramagnetic relaxation is outlined.

Quantum channels (CPTP map)

$$\dim \mathcal{H} = d < \infty$$

$$\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

- completely positive
- trace-preserving

$$\Phi(X) = \sum_\alpha K_\alpha X K_\alpha^\dagger$$

$$\sum_\alpha K_\alpha^\dagger K_\alpha = \mathbb{1}$$

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$$\Phi(X) = \sum_{\alpha} S_\alpha X S_\alpha^\dagger$$

$$\sum_{\alpha} S_\alpha^\dagger S_\alpha = \mathbb{1}$$

$$B_{rr',ss'} = \sum_{q,q'} \mu(qq') W_{rr'}(qq') (W_{ss'}(qq'))^*. \qquad (32)$$

The strong trace relation (17) ensures that the relation

$$\sum \mu(qq') W_{rs}(qq') W_{sr'}{}^\dagger (qq') = \delta_{rr'}, \qquad (33)$$

will be valid. These results are stated in the following theorem:

Theorem 4. A general dynamical matrix can be written in the canonical form (32) in terms of n^2 matrices $W(qq')$ which obey the bilinear relations (31) and (33).

Note that the matrices $W(qq')$ are not necessarily unitary, but satisfy only the weaker condition (33). In

Dynamical semigroup

$$\Lambda_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

- Λ_t is CPTP for $t \geq 0$
- $\Lambda_0 = \mathrm{id}$
- $\Lambda_{t+s} = \Lambda_t \, \Lambda_s$

$$\Lambda_t = e^{t\mathcal{L}} \; ; \qquad \mathcal{L} \; = ???$$

Completely positive dynamical semigroups of N-level systems*

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(Received 19 March 1975)

We establish the general form of the generator of a completely positive dynamical semigroup of an N-level quantum system, and we apply the result to derive explicit inequalities among the physical parameters characterizing the Markovian evolution of a 2-level system.

I. INTRODUCTION

In this paper we establish the general form of the generator of a completely positive dynamical semigroup of an N-level quantum system (Sec. II) and we find the conditions, in the form of explicit inequalities, that complete positivity imposes on the physical param-

the gme does indeed go over into an equation of the form (1.1) with a rescaled time variable in the limit when the coupling of the system to its surroundings is made to tend to zero (weak-coupling limit, $\tau_S \to \infty$).[20] It is also possible to obtain (1.1) rigorously in the lim $\tau_R \to 0$. This has been called the limit of *singular reservoir*.[21] See our next paper for an explicit model

Markovian semigroup

$$\dim \mathcal{H} = d < \infty \;; \quad \hbar = 1$$

$$\Lambda_t = e^{t\mathcal{L}} \quad \longleftrightarrow \quad \frac{d}{dt}\Lambda_t = \mathcal{L}\Lambda_t$$

Theorem (Gorini-Kossakowski-Sudarshan-Lindblad (1976))

$\Lambda_t = e^{t\mathcal{L}}$ *is CPTP for* $t \geq 0$ *if and only if*

$$\mathcal{L}(\rho) \;=\; -i[H,\rho] + \mathcal{L}_D(\rho)$$

$$\mathcal{L}_D(\rho) = \sum_k \gamma_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right) \;\; ; \;\; \gamma_k > 0$$

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PROPERTIES OF QUANTUM MARKOVIAN MASTER EQUATIONS*

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(*Received January 5, 1977*)

Andrzej Kossakowski (1938-2021)

40 years after (Toruń 2016)

$$\mathcal{L}(\rho) \ = \ -i[H,\rho] + \mathcal{L}_D(\rho)$$

$$\mathcal{L}_D(\rho) = \sum_k \gamma_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right) \quad ; \quad \gamma_k > 0$$

- the representation is not unique
- even the splitting is not unique
- the rates γ_k depend upon the representation (and hence they are not directly measured)

$$\mathcal{L}(\rho) \;=\; -i[H,\rho] + \mathcal{L}_D(\rho)$$

$$\mathcal{L}_D(\rho) = \sum_k \gamma_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right) \quad ; \quad \gamma_k > 0$$

Complete positivity $\longrightarrow \begin{cases} \text{Math} & \gamma_k \geq 0 \\ \text{Phys} & ??? \end{cases}$

$$\mathcal{L}(\rho) \;=\; -i[H,\rho] + \mathcal{L}_D(\rho)$$

$$\mathcal{L}_D(\rho) = \sum_k \gamma_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right) \quad ; \quad \gamma_k > 0$$

Complete positivity $\longrightarrow$ $\begin{cases} \text{Math} & \gamma_k \geq 0 \\ \text{Phys} & \text{SPECTRUM} \end{cases}$

Spectrum does not depend on the representation.

Spectra of generators

$$\mathcal{L}(X_\alpha) = \ell_\alpha X_\alpha \quad (\Rightarrow \quad \mathcal{L}(X_\alpha^\dagger) = \ell_\alpha^* X_\alpha^\dagger)$$

Theorem

- $\ell \in \mathrm{spec}(\mathcal{L}) \quad \Leftrightarrow \quad \ell^* \in \mathrm{spec}(\mathcal{L})$
- *there is a leading eigenvalue* $\ell_0 = 0$
- *the corresponding eigenvector defines a density operator*
- $\mathrm{Re}\,\ell_\alpha \leq 0$ *for* $\alpha = 1, 2, \ldots, d^2 - 1$.

$$\Gamma_\alpha := -\mathrm{Re}\,\ell_\alpha \quad \text{(relaxation rates)}$$

$$T_\alpha := 1/\Gamma_\alpha \quad \text{(relaxation times)}$$

γ_α vs Γ_α

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_k \gamma_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right) \quad ; \quad \gamma_k > 0$$

Positivity of Γ_α is necessary but NOT sufficient for CP

Example: GKS 1976

$$\mathcal{L}(\rho) = -i\frac{\omega}{2}[\sigma_z, \rho] + \gamma_+\mathcal{L}_+ + \gamma_-\mathcal{L}_- + \gamma_z\mathcal{L}_z$$

$$\begin{aligned}
\mathcal{L}_+(\rho) &= \sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho\} \\
\mathcal{L}_-(\rho) &= \sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\} \\
\mathcal{L}_z(\rho) &= \sigma_z\rho\sigma_z - \rho
\end{aligned}$$

$$\Gamma_\mathrm{T} := \Gamma_1 = \Gamma_2 = \frac{1}{2}(\gamma_+ + \gamma_-) + \gamma_z \; ; \quad \Gamma_\mathrm{L} = \Gamma_3 = \gamma_+ + \gamma_-$$

$$T_\mathrm{L} \geq 2\,T_\mathrm{T}$$

Bloch equations

$$M_k := \mathrm{Tr}(\rho\sigma_k)\ ; \quad k = x, y, z$$

$$\begin{aligned}
\dot{M}_x &= \omega M_y - \frac{M_x}{T_\mathrm{T}} \\
\dot{M}_y &= -\omega M_x - \frac{M_y}{T_\mathrm{T}} \\
\dot{M}_z &= -\frac{M_z - M_0}{T_\mathrm{L}}
\end{aligned}$$

$$T_\mathrm{L} \geq 2\, T_\mathrm{T}$$

To stay within a Bloch ball it is sufficient to have $T_\mathrm{L}, T_\mathrm{T} > 0$

Example: GKS 1976

$$\Gamma_{\mathrm{T}} := \Gamma_1 = \Gamma_2 = \frac{1}{2}(\gamma_+ + \gamma_-) + \gamma_z \; ; \quad \Gamma_{\mathrm{L}} = \Gamma_3 = \gamma_+ + \gamma_-$$

$$\Gamma := \Gamma_1 + \Gamma_2 + \Gamma_3 = 2(\gamma_+ + \gamma_- + \gamma_z)$$

$$\Gamma_k \leq \frac{1}{2}\Gamma \quad (k = 1, 2, 3)$$

General qubit Lindbladian

G. Kimura, PRA 2002

$$\Gamma := \Gamma_1 + \Gamma_2 + \Gamma_3$$

$$\Gamma_k \leq \frac{1}{2}\Gamma \quad (k = 1, 2, 3)$$

$$R_k := \frac{\Gamma_k}{\Gamma}$$

$$R_k \leq \frac{1}{2} \quad (k = 1, 2, 3)$$

general d

Wolf & Cirac, CMP (2008): $\mathcal{L} = \mathcal{L}_D$

$$\|\mathcal{L}\|_\infty \leq \frac{2}{d}\,\Gamma$$

$$\Gamma_k = -\mathrm{Re}\,\ell_k \implies \Gamma_k \leq \frac{2}{d}\,\Gamma$$

Kimura et al, OSID (2017): arbitrary $\mathcal{L}$

$$\Gamma_k \leq \frac{\sqrt{2}}{d}\,\Gamma$$

$$\Gamma_k \leq \frac{c_d}{d}\,\Gamma$$

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$$\Gamma_k \leq \frac{\sqrt{2}}{d}\,\Gamma$$

$$\Gamma_k \leq \frac{c_d}{d}\,\Gamma$$

Please wait for Gen Kimura talk!

Conjecture: $c_d = 1 \quad \longrightarrow \quad \Gamma_k \leq \frac{1}{d}\Gamma$

Please wait for Gen Kimura talk!

Conjecture: $c_d = 1 \quad \longrightarrow \quad \Gamma_k \leq \frac{1}{d}\,\Gamma$

Conjecture

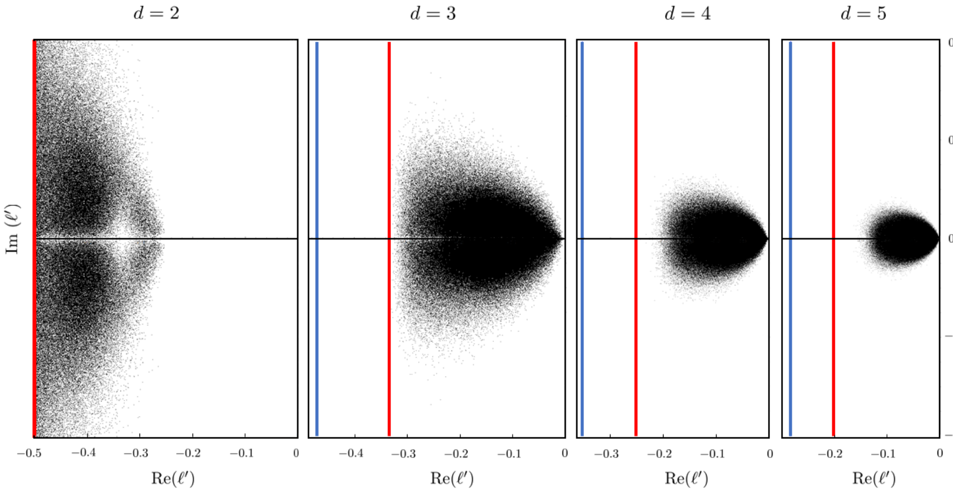
d-level Lindbladian

$$\Gamma := \Gamma_1 + \ldots + \Gamma_{d^2-1}$$

$$\Gamma_k \leq \frac{1}{d}\Gamma \quad (k = 1, \ldots, d^2 - 1)$$

$$R_k := \frac{\Gamma_k}{\Gamma}$$

$$R_k \leq \frac{1}{d} \quad (k = 1, \ldots, d^2 - 1)$$



Distribution of ℓ_α/Γ for random Lindbladians. Red vertical lines denote the bound '$-1/d$'. Blue vertical lines denote the bound '$-\sqrt{2}/d$'.

PHYSICAL REVIEW LETTERS **123,** 140403 (2019)

Universal Spectra of Random Lindblad Operators

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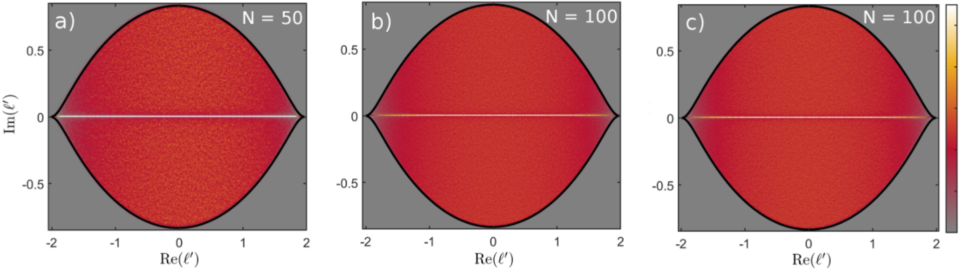
To understand the typical dynamics of an open quantum system in continuous time, we introduce an ensemble of random Lindblad operators, which generate completely positive Markovian evolution in the space of the density matrices. The spectral properties of these operators, including the shape of the eigenvalue distribution in the complex plane, are evaluated by using methods of free probabilities and explained with non-Hermitian random matrix models. We also demonstrate the universality of the spectral features. The notion of an ensemble of random generators of Markovian quantum evolution constitutes a step towards categorization of dissipative quantum chaos.

Lemon-like shape

$$N(\mathcal{L}+1) \qquad (N=d)$$



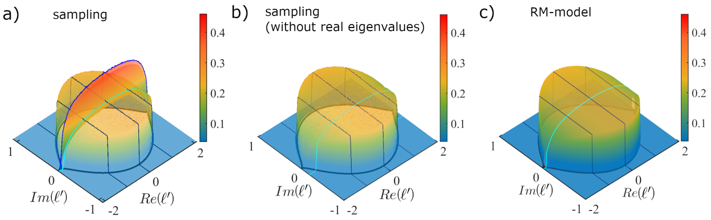
The boundary of the spectrum of the łemon"is given by the solution of equation for complex z

$$\mathsf{Im}[z + G(z)] = 0, \quad \text{with}$$

$$G(z) = 2z - \frac{2z}{3\pi}\left[(4+z^2)E\left(\frac{4}{z^2}\right) + (4-z^2)K\left(\frac{4}{z^2}\right)\right],$$

$E(k)$ and $K(k)$ are complete elliptic integrals of the first and second kind.

Spectral density



$$z = x + iy \; ; \quad \rho(x, y) = \rho(x)$$

$$\Gamma_k \leq \frac{1}{d}\,\Gamma$$

- is tight
- **true** for unital semigroups
- **true** for covariant semigroups
- **true** for Davies generators (weak coupling limit)
- implications

PHYSICAL REVIEW LETTERS **127,** 050401 (2021)

Universal Constraint for Relaxation Rates for Quantum Dynamical Semigroup

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A general property of relaxation rates in open quantum systems is discussed. We find an interesting constraint for relaxation rates that universally holds in fairly large classes of quantum dynamics, e.g., weak coupling regimes, as well as for entropy nondecreasing evolutions. We conjecture that this constraint is universal, i.e., it is valid for all quantum dynamical semigroups. The conjecture is supported by numerical analysis. Moreover, we show that the conjectured constraint is tight by providing a concrete model that saturates the bound. This universality marks an essential step toward the physical characterization of complete positivity as the constraint is directly verifiable in experiments. It provides, therefore, a physical manifestation of complete positivity. Our conjecture also has two important implications: it provides (i) a universal constraint for the spectra of quantum channels and (ii) a necessary condition to decide whether a given channel is consistent with Markovian evolution.

Schrödinger vs. Heisenberg

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_k \gamma_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right)$$

$$\mathcal{L}^\ddagger(X) = i[H,X] + \sum_k \gamma_k \left(L_k^\dagger X L_k - \frac{1}{2}\{L_k^\dagger L_k, X\} \right)$$

$$\mathrm{Tr}(X\mathcal{L}^\ddagger(Y)) := \mathrm{Tr}(\mathcal{L}(X)Y)$$

Schrödinger vs. Heisenberg

$$\mathcal{L}(X_\alpha) = \ell_\alpha X_\alpha$$

$$\mathcal{L}^\ddagger(Y_\alpha) = \ell_\alpha^* Y_\alpha$$

$$\mathcal{L}(\rho^{\mathrm{eq}}) = 0$$

$$\mathcal{L}^\ddagger(\mathbb{1}) = 0$$

$$\mathcal{L}(\rho) \;=\; -i[H,\rho] + \sum_k \gamma_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right)$$

$$\rho^{\mathrm{eq}} > 0 \;, \quad (A,B)_{\mathrm{eq}} := \mathrm{Tr}(\rho^{\mathrm{eq}} A^\dagger B) \quad \longrightarrow \quad \|A\|_{\mathrm{eq}}^2 = (A,A)_{\mathrm{eq}}$$

$$\Gamma_\alpha = \frac{1}{2\|Y_\alpha\|_{\mathrm{eq}}^2} \sum_k \gamma_k \|[L_k, Y_\alpha]\|_{\mathrm{eq}}^2$$

Unital semigroup

$$\rho^{\text{eq}} = \frac{1}{d}\,\mathbb{1}$$

$$\mathcal{L}(\mathbb{1}) = 0$$

$$\Gamma_\alpha = \frac{1}{2\|Y_\alpha\|^2_{\text{eq}}} \sum_k \gamma_k \|[L_k, Y_\alpha]\|^2_{\text{eq}}$$

$$\Gamma_\alpha = \frac{1}{2\|Y_\alpha\|^2} \sum_k \gamma_k \|[L_k, Y_\alpha]\|^2 \quad ; \quad \|A\|^2 = \text{Tr}(A^\dagger A)$$

Unital semigroup

$$\Gamma_\alpha = \frac{1}{2\|Y_\alpha\|^2} \sum_k \gamma_k \|[L_k, Y_\alpha]\|^2$$

Bötcher & Wenzel, Lin. Alg. Appl. (2008)

$$\|[A, B]\|^2 \leq 2\|A\|^2 \|B\|^2$$

$$\Gamma_\alpha \leq \frac{1}{d}\Gamma$$

Covariant generators

$$U_{\mathbf{x}} \mathcal{L}(X) U_{\mathbf{x}}^{\dagger} = \mathcal{L}(U_{\mathbf{x}} X U_{\mathbf{x}}^{\dagger})$$

$$U_{\mathbf{x}} = \sum_{k=1}^{d} e^{-ix_k} |k\rangle\langle k| \quad ; \quad \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$$

$$\Gamma_\alpha \leq \frac{1}{d}\Gamma$$

Weak coupling limit — Davies generators

$$\mathcal{L}(\rho) = -i[H, \rho] + \mathcal{L}_D(\rho)$$

- $H = \sum_i h_i |i\rangle\langle i|$
- $\rho^{\text{eq}} = \sum_i p_i |i\rangle\langle i|$
- $\mathcal{L}_D^{\ddagger}$ is Hermitian w.r.t. $(\;,\;)_{\text{eq}}$

$$\Gamma_\alpha \leq \frac{1}{d} \Gamma$$

Implications

$$\Gamma_\alpha \leq \frac{1}{d}\Gamma$$

- spectra of channels
- is quantum channel Φ Markovian, that is, $\Phi = e^{\mathcal{L}}$?

Spectra of channels: $\Gamma_\alpha \leq \frac{1}{d}\Gamma$

$$\mathcal{L} = \Phi - \mathrm{id}$$

Theorem

The spectrum $z_\alpha = x_\alpha + iy_\alpha$ of any unital quantum channel satisfy

$$\sum_{\beta=1}^{d^2-1} x_\beta \leq d(d-1) - 1 + dx_\alpha,$$

for $\alpha = 1, \ldots, d^2 - 1$.

Theorem

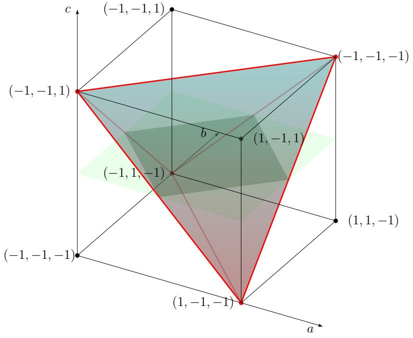
The spectrum $z_\alpha = x_\alpha + iy_\alpha$ of any qubit channel satisfies

$$|x_1 \pm x_2| \leq 1 \pm x_3.$$

Pauli qubit channels: $\Phi(\rho) = \sum_{\alpha=1}^{3} p_\alpha \sigma_\alpha \rho \sigma_\alpha$

Spectrum $= \{1, \lambda_1, \lambda_2, \lambda_3\}$

Algoet – Fujiwara : $|\lambda_1 \pm \lambda_2| \leq 1 \pm \lambda_3 \longleftrightarrow |x_1 \pm x_2| \leq 1 \pm x_3$



Markovianity $\Gamma_\alpha \leq \frac{1}{d}\Gamma$

M. Wolf, et al PRL (2008).
Unital Φ. Is $\Phi = e^{\mathcal{L}}$?

$$\mathcal{L} \quad \longrightarrow \quad \ell_\alpha = iy_\alpha - \Gamma_\alpha$$

$$\Phi \quad \longrightarrow \quad z_\alpha = e^{\ell_\alpha}$$

$$\det \Phi = z_1 \dots z_{d^2-1} = e^{-\Gamma} \leq e^{d\Gamma_\alpha} = |z_\alpha|^d$$

$$\sqrt[d]{\det \Phi} \leq |z_\alpha| \leq 1$$

Frobenius-Perron ring

Example: qubit Pauli channel

$$z_k = e^{-\Gamma_k}$$

$$z_1 z_2 z_3 \leq z_k^2 \quad (k = 1, 2, 3)$$

$$z_1 z_2 \leq z_3 \;, \quad \text{etc.}$$

Davalos et al, Quantum (2019)
Puchała et al, Phys. Lett. A (2019)

Conlusions

- For any qubit Lindbladian $\Gamma_k \leq \frac{1}{2}\Gamma$
- For any unital Lindbladian $\Gamma_k \leq \frac{1}{d}\Gamma$
- True for any 'physical generator' obtained in the weak coupling limit
- The conjecture for arbitrary $\mathcal{L}$ is supported by numerical analysis
- New bounds for the spectra of channels
- Necessary condition for Markovianity (for unital channels) $\sqrt[d]{\det\Phi} \leq |z_\alpha| \leq 1$
- D.C., G. Kimura, A. Kossakowski, and Y. Shishido, PRL 2021
- D.C R. Fujii, G. Kimura, and H. Ohno, Lin. Alg. App. 2021

Part II