Dual and multiunitary operators,from entanglement to many-body physics, as a legacy of the SMR paper

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STOCHASTIC DYNAMICS OF QUANTUM MECHANICAL SYSTEMS
WARSAW, OCT. 14, 2021

malized weights the most general dynamical law is given by the linear, homogeneous mapping:

$$\rho_{r,s}(t_0) \longrightarrow \rho_{r,s}(t) = A_{rs,r's'}(t,t_0)\rho_{r's'}(t_0), \qquad (14)$$

where $A_{rs,r's'}(t,t_0)$ is a numerical $n^2 \times n^2$ matrix labelled by pairs of indices (rs) and $(r's')$ depending on the times t and t_0 but *independent* of the matrix $\rho(t_0)$. Since the

matrix. To display these properties in a more transparent fashion, as well as for further development, it is advantageous to introduce another $n^2 \times n^2$ matrix B related to A and defined by

$$B_{rr',ss'} = A_{rs,r's'}. \qquad (14)$$

It immediately follows that B is Hermitian and positive semidefinite; we can rewrite (11′) and (12′) in the form:

Sudarshan, Mathews, and Rau, Phys. Rev. (1961)

Vectorization (row-major, see Wikipedia)

vec(A) : $\langle i|A|\alpha\rangle = \langle i\alpha|A\rangle$, vec$(ABC) = (A \otimes B^T)vec(C)$

Unitary quantum evolution in $\mathcal{H}^n$

$\rho' = U\rho U^\dagger$, $|\rho'\rangle = (U \otimes U^*)|\rho\rangle = L|\rho\rangle$, **Superoperator**: $L = U \otimes U^*$

Realignment: $\langle ij|L^R|\alpha\beta\rangle = \langle i\alpha|L|j\beta\rangle$

$= \langle i\alpha|(U \otimes U^*)|j\beta\rangle = \langle i|U|j\rangle\langle\alpha|U^*|\beta\rangle = \langle ij|U\rangle\langle U|\alpha\beta\rangle$

Dynamical matrix: $L^R = |U\rangle\langle U|$, $(A \otimes B)^R = |A\rangle\langle B^*|$

General "CPTP" quantum evolution in $\mathcal{H}^n$

positive definite $L^R = \sum_m \lambda_m|\lambda_m\rangle\langle\lambda_m|$, $L = \sum_m \lambda_m \Lambda_m \otimes \Lambda_m^*$

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TABLE II: Quantum operations $\Phi : \mathcal{M}^{(N)} \to \mathcal{M}^{(N)}$: properties of superoperator L and dynamical matrix $D_\Phi = L^R$

Matrices	Superoperator $L = D^R$	Dynamical matrix D_Φ
Hermicity	No	Yes
Trace	spectrum is symmetric $\Rightarrow \quad \mathrm{tr}L \in \mathbb{R}$	$\mathrm{tr}D_\Phi = N$
(right) Eigenvectors	invariant states or transient corrections	Kraus operators
Eigenvalues	$\|z_i\| \leq 1$ $-\ln\|z_i\|$ – decay rates	weights of Kraus operators, $d_i \geq 0$
Unitary evolution $D_\Phi = (U \otimes U^*)^R$	$\|\|L\|\|_2 = N$	$S(\vec{d}) = 0$
Coarse graining	$\|\|L\|\|_2 = \sqrt{N}$	$S(\vec{d}) = \ln N$
Complete depolarisation, $D_\Phi = \mathbb{1}$	$\|\|L\|\|_2 = 1$	$S(\vec{d}) = 2\ln N$

Completely depolarizing channel: $\rho' = I/N$ maximally mixed.
$L^R = I/N$, $L = |\Phi^+\rangle\langle\Phi^+|$, $|\Phi^+\rangle = \sum_m |mm\rangle/\sqrt{N}$ maximally entangled.

From **On duality between quantum maps and quantum states**
Karol Życzkowski and Ingemar Bengtsson (2004)

Realignment, Reshuffle

X: matrix in $\mathcal{H}^N \otimes \mathcal{H}^N$: $N^2 \times N^2$ matrix.
Row and col. label: (i,α), $1 \le i,\alpha \le N$.

$$\langle i\alpha|X|j\beta\rangle = \langle \beta\alpha|X^{R_1}|ji\rangle$$

Example $N = 2$:

$$X = \left(\begin{array}{cc|cc} X_{00,00} & X_{00,01} & X_{00,10} & X_{00,11} \\ X_{01,00} & X_{01,01} & X_{01,10} & X_{01,11} \\ \hline X_{10,00} & X_{10,01} & X_{10,10} & X_{10,11} \\ X_{11,00} & X_{11,01} & X_{11,10} & X_{11,11} \end{array}\right)$$

$$X^{R_1} = \left(\begin{array}{c|c|c|c} X_{00,00} & X_{10,00} & X_{00,10} & X_{10,10} \\ X_{01,00} & X_{11,00} & X_{01,10} & X_{11,10} \\ X_{00,01} & X_{10,01} & X_{00,11} & X_{10,11} \\ X_{01,01} & X_{11,01} & X_{01,11} & X_{11,11} \end{array}\right)$$

Realignment, Reshuffle, Partial Transposes

$$X^{R_2} = \left(\begin{array}{cccc} X_{00,00} & X_{00,01} & X_{01,00} & X_{01,01} \\ \hline X_{00,10} & X_{00,11} & X_{01,10} & X_{01,11} \\ \hline X_{10,00} & X_{10,01} & X_{11,00} & X_{11,01} \\ \hline X_{10,10} & X_{10,11} & X_{11,10} & X_{11,11} \end{array}\right)$$

$$X^{T_1} = \left(\begin{array}{cc|cc} X_{00,00} & X_{00,01} & X_{10,00} & X_{10,01} \\ X_{01,00} & X_{01,01} & X_{11,00} & X_{11,01} \\ \hline X_{00,10} & X_{00,11} & X_{10,10} & X_{10,11} \\ X_{01,10} & X_{01,11} & X_{11,10} & X_{11,11} \end{array}\right)$$

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Why these permutations?

$$\text{Swap} \ \ S|i\alpha\rangle = |\alpha i\rangle$$

$$(A^{T_1})^{T_2} = (A^{T_2})^{T_1} = A^T, \ (A^{T_1})^T = (A^T)^{T_1},$$
$$SA^{T_1}S = (SAS)^{T_2}, \ SA^{T_2}S = (SAS)^{T_1},$$
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Effect of local operators

$$[(u_1 \otimes u_2)A(u_3 \otimes u_4)]^{R_1} = (u_4^T \otimes u_2)A^{R_1}(u_3 \otimes u_1^T),$$
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Entanglement and these permutations

$\rho \in \mathcal{H}^N \otimes \mathcal{H}^N$ Bipartite state: **Separable iff** $\rho = \sum_k p_k \rho_k^A \otimes \rho_k^B$.

Peres Partial Transpose criterion

State is entangled if $\rho^{T_1} < 0$: Negative partial transpose (NPT) state, else PPT. (Peres, 1996)

Cross-Norm/ Realignment criterion

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Spacetime Duals and realignment of unitaries

where we introduced the dual local gate $\tilde{U}$ by means of the following reshuffling:

$$\langle k| \otimes \langle \ell|\tilde{U}|i\rangle \otimes |j\rangle = \langle j| \otimes \langle \ell|U|i\rangle \otimes |k\rangle. \qquad (5)$$

The dual gate defines the evolution in a circuit where the roles of time and space have been swapped.

from "Exact Correlation Functions for Dual-Unitary Lattice Models in 1 + 1 Dimensions",

Bruno Bertini , Pavel Kos, and Tomaz Prosen, PRL, 2019

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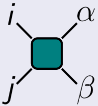
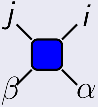
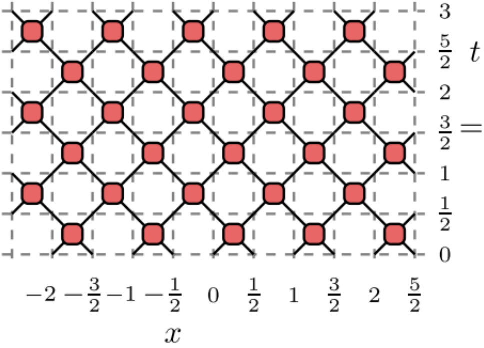
$\langle i\alpha|U|j\beta\rangle =$ 

$=$ $\langle \beta\alpha|U^{R_1}|ji\rangle =$ 

Spacetime duality in unitary circuits

$$\mathrm{tr}\left[\mathbb{U}^t\right] = \quad (\text{diagram}) \quad = \mathrm{tr}\left[\tilde{\mathbb{U}}^L\right].$$



$2L$ particles, propagator U, time t

$\equiv$

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U is dual-unitary if Realigned U is also unitary

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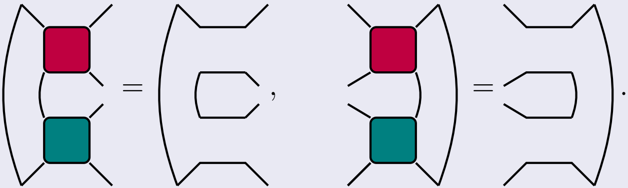

$\iff U$ has maximum operator entanglement $E(U)$

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$$E(U)=1-\frac{1}{N^4}\sum_{i=1}^{N^2}\lambda_i^2,\quad \max E(U)=1-\frac{1}{N^2}=E(S)$$

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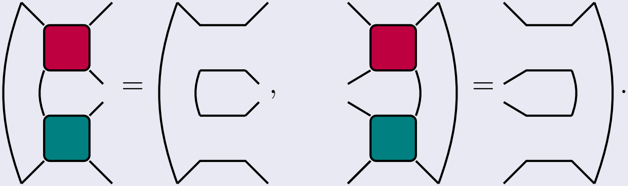

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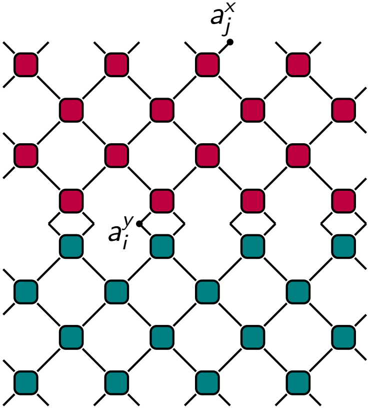
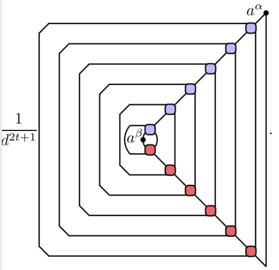
Correlations in dual-unitary circuits

$$D^{ij}(x,y,t) \equiv \frac{1}{q^{2L}} \mathrm{tr}[a_j^x \mathbb{U}^{-t} a_i^y \mathbb{U}^t] =$$

$\frac{1}{q^{2L}}$



Correlations on the lightcone



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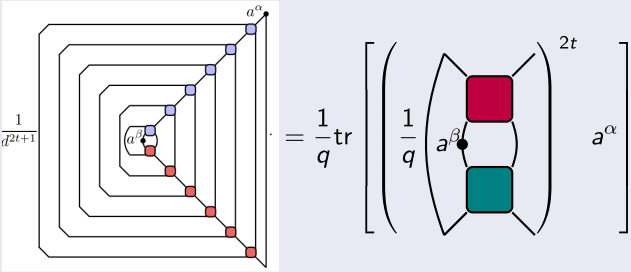
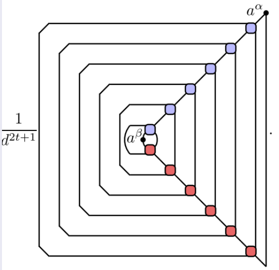
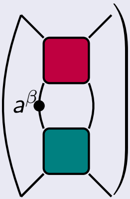


Correlations on the lightcone

$\frac{1}{d^{2t+1}}$



$= \frac{1}{q}\mathrm{tr}\Bigg[\Bigg(\frac{1}{q}$



$\Bigg)^{2t} a^\alpha\Bigg] = \frac{1}{q}\mathrm{tr}[\mathcal{M}_+^{2t}(a^\beta)\, a^\alpha]$

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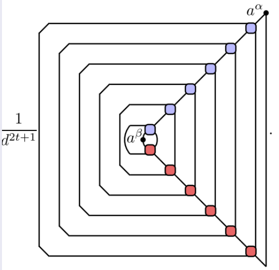

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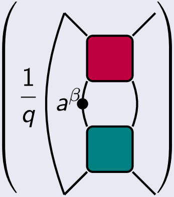
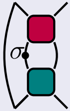


$$\left.\left.\right)^{2t} a^\alpha\right] = \frac{1}{q}\mathrm{tr}[\mathcal{M}_+^{2t}(a^\beta)\, a^\alpha]$$

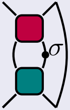
The CPTP maps

$$\mathcal{M}_+(\sigma) = \frac{1}{q}\mathrm{tr}_1\left[U^\dagger(\sigma\otimes\mathbb{1})U\right] = \frac{1}{q}$$



$$M_+[U] = \frac{1}{q}\left(U^{T_2}U^{T_2\dagger}\right)^{R_1}$$

$$\mathcal{M}_-(\sigma) = \frac{1}{q}\mathrm{tr}_2\left[U^\dagger(\mathbb{1}\otimes\sigma)U\right] = \frac{1}{q}$$



$$M_-[U] = \frac{1}{q}\left(U^{T_2\dagger}U^{T_2}\right)^{R_2}$$

Eigenvalues of the unital contractive maps

- $\text{Spec}\{\mathcal{M}_{\pm}\} = \{\lambda_j^{\pm},\ 0 \leq j \leq q^2 - 1\}$
- $\lambda_0^{\pm} = 1$ corresponding state is the identity operator: trivial eigenvalue.
- $1 \geq |\lambda_1^{\pm}| \geq \cdots \geq |\lambda_{q^2-1}^{\pm}|$

determines the correlation functions

$$C_{\pm}^{\alpha\beta}(x = \pm t, t) = \sum_{j=1}^{q^2-1} C_j^{\alpha\beta} (\lambda_j^{\pm})^{2t}$$

Ergodic classification of dual-circuits

The nontrivial eigenvalues $\lambda_k^{\pm}$ ($k > 0$) determines ergodic properties.

- **Non-interacting:** All $2(q^2-1)$ eigenvalues $\lambda_k = 1$, the correlations remain constant in all modes and the system is non-interacting.

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- **Ergodic and mixing:** All $|\lambda_k| < 1$, the correlations decay exponentially.

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- **Ergodic and mixing:** All $|\lambda_k| < 1$, the correlations decay exponentially.
- **Bernoulli:** For $k > 0$, $\lambda_k = 0$. Correlations decay instantly, $M_{\pm}$ **completely depolarizing channels** (Suhail A Rather, Aravinda S, AL, Physical Review Research (To appear, 2021))

2-unitary operators, AME and perfect tensors

$$U,\ U^{R_2},\ U^{T_2}\ \in \mathcal{U}(q^2)$$

$$\sum_{i\alpha j\beta=1}^{q} U_{i\alpha j\beta}|i\alpha j\beta\rangle$$

is an Absolutely Maximally Entangled state of 4 parties: AME$(4, q)$.

(Dardo Goyeneche, Daniel Alsina, Jose I. Latorre, A. Riera, and Karol Zyczkowski, "Absolutely maximally ˙ entangled states, combinatorial designs, and multiunitary matrices," Phys. Rev. A 92, 032316 (2015))

- 2-unitary operators maximize entangling power.
- There are NO 2-unitaries for $q = 2$, Qubits. No AME$(4, 2)$. (Sudbery, Higuchi).
- 2-unitary circuits are Bernoulli: 2-body operators also decorrelate instantly. (Suhail A Rather, Aravinda S, AL, Physical Review Research (To appear, 2021))

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The art of constructing dual unitary and 2-unitary/AME

- Based on block diagonal and swap combinations.

PHYSICAL REVIEW LETTERS **125,** 070501 (2020)

Creating Ensembles of Dual Unitary and Maximally Entangling Quantum Evolutions

Suhail Ahmad Rather,* S. Aravinda,† and Arul Lakshminarayan‡
Department of Physics, Indian Institute of Technology Madras, Chennai 600036, India

(Received 14 February 2020; accepted 15 July 2020; published 10 August 2020)

Maximally entangled bipartite unitary operators or gates find various applications from quantum information to many-body physics wherein they are building blocks of minimal models of quantum chaos. In the latter case, they are referred to as "dual unitaries." Dual unitary operators that can create the maximum average entanglement when acting on product states have to satisfy additional constraints. These have been called "2-unitaries" and are examples of perfect tensors that can be used to construct absolutely maximally entangled states of four parties. Hitherto, no systematic method exists in any local dimension, which results in the formation of such special classes of unitary operators. We outline an iterative protocol, a nonlinear map on the space of unitary operators, that creates ensembles whose members are arbitrarily close to being dual unitaries. For qutrits and ququads we find that a slightly modified protocol yields a plethora of 2-unitaries.

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- Nonlinear maps on $U(q^2)$.

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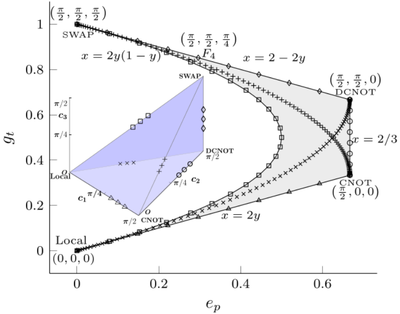
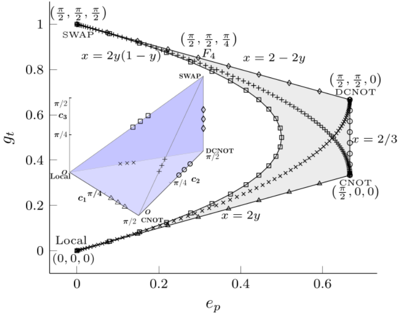
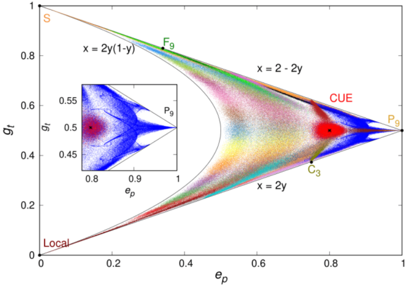
Qubit and Qutrit gates



From "Entanglement measures of bipartite quantum gates and their thermalization under arbitrary interaction strength" Bhargavi Jonnadula, Prabha Mandayam, Karol Życzkowski, AL, Phys. Rev. Res. 2020





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QuNits/Qudits, $d > 3$

2$-$unitary gates exists for all $U(N^2)$, $N > 2$

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Solves the puzzle of if there are absolutely maximally entangled states AME$(4, 6)$.

The golden AME(4, 6) state

(1,1)	(2,2)	(1,2)	(2,1)	
$a\,\omega^{10}$	a	$b\,\omega^{15}$	$b\,\omega^{5}$	(6,3)
		c	c	(1,1)
c	c			(5,6)
$b\,\omega^{10}$	b	$a\,\omega^{5}$	$a\,\omega^{15}$	(4,2)

(1,3)	(2,4)	(1,4)	(2,3)	
$a\,\omega^{2}$	$a\,\omega^{14}$	$b\,\omega$	$b\,\omega^{5}$	(2,5)
		$c\,\omega^{5}$	$c\,\omega^{19}$	(3,3)
$c\,\omega^{17}$	$c\,\omega^{19}$			(1,2)
$b\,\omega^{14}$	$b\,\omega^{6}$	$a\,\omega^{3}$	$a\,\omega^{7}$	(6,4)

(1,5)	(2,6)	(1,6)	(2,5)	
$a\,\omega$	$a\,\omega^{19}$	$b\,\omega^{14}$	$b\,\omega^{16}$	(4,1)
$a\,\omega$	$a\,\omega^{3}$	$b\,\omega^{10}$	$b\,\omega^{4}$	(3,4)
$b\,\omega^{4}$	$b\,\omega^{18}$	$a\,\omega^{3}$	$a\,\omega^{9}$	(2,6)
$b\,\omega^{2}$	$b\,\omega^{8}$	$a\,\omega^{5}$	$a\,\omega^{15}$	(5,5)

(3,1)	(4,2)	(3,2)	(4,1)	
$a\,\omega^{4}$	$a\,\omega^{10}$	$b\,\omega^{17}$	$b\,\omega^{7}$	(4,5)
		$c\,\omega^{2}$	$c\,\omega^{2}$	(3,2)
$c\,\omega^{10}$	$c\,\omega^{6}$			(2,4)
$b\,\omega^{7}$	$b\,\omega^{13}$	$a\,\omega^{10}$	a	(5,3)

(3,3)	(4,4)	(3,4)	(4,3)	
a	a	$b\,\omega^{15}$	$b\,\omega^{15}$	(4,6)
		c	$c\,\omega^{10}$	(6,1)
c	$c\,\omega^{10}$			(5,4)
b	b	$a\,\omega^{5}$	$a\,\omega^{5}$	(1,5)

(3,5)	(4,6)	(3,6)	(4,5)	
$a\,\omega^{2}$	a	$b\,\omega^{19}$	$b\,\omega^{13}$	(2,3)
		$c\,\omega^{16}$	c	(6,2)
$c\,\omega^{8}$	$c\,\omega^{16}$			(3,1)
$b\,\omega^{14}$	$b\,\omega^{12}$	$a\,\omega$	$a\,\omega^{15}$	(1,6)

(5,1)	(6,2)	(5,2)	(6,1)	
$a\,\omega^{3}$	$a\,\omega^{7}$	b	b	(1,4)
		c	$c\,\omega^{10}$	(2,1)
$c\,\omega^{13}$	$c\,\omega^{7}$			(3,5)
$b\,\omega^{9}$	$b\,\omega^{13}$	$a\omega^{16}$	$a\omega^{16}$	(6,6)

(5,3)	(6,4)	(5,4)	(6,3)	
$a\,\omega^{12}$	$a\,\omega^{14}$	$b\,\omega^{15}$	$b\,\omega$	(3,6)
		$c\,\omega^{14}$	$c\,\omega^{10}$	(5,1)
$c\,\omega^{7}$	$c\,\omega^{19}$			(2,2)
$b\,\omega^{14}$	$b\,\omega^{16}$	$a\omega^{7}$	$a\omega^{13}$	(4,3)

(5,5)	(6,6)	(5,6)	(6,5)	
$a\,\omega^{18}$	$a\,\omega^{18}$	$b\,\omega^{3}$	$b\,\omega^{3}$	(1,3)
		c	$c\,\omega^{10}$	(5,2)
$c\,\omega$	$c\,\omega^{11}$			(6,5)
$b\,\omega^{10}$	$b\,\omega^{10}$	$a\omega^{5}$	$a\omega^{5}$	(4,4)

AME(4,6) state

$$\frac{1}{6}\sum_{i,j,k,\ell=1}^{d} t_{i,j,k,\ell}\,|i\rangle\,|j\rangle\,|k\rangle\,|\ell\rangle$$

$$a = \frac{1}{\sqrt{2}(\omega+\overline{\omega})} = \frac{1}{\sqrt{5+\sqrt{5}}}$$

$$b = \frac{1}{\sqrt{2}(\omega^3+\overline{\omega}^3)} = \sqrt{\frac{5+\sqrt{5}}{20}}$$

$$c = \frac{1}{\sqrt{2}},\quad \omega = e^{2\pi i/20}$$

All U, U^{R_1} and U^{T_1} are unitary.

Last slide

- The SMR paper continues to stimulate our understanding in quantum information theory, many-body physics, open systems, quantum chaos and much else.
- While "Dual operators" played a central role in this talk, ECG was a **nondualist**!
- Sudarshan was most taken up by the problem of irreversibility and time: *"There are certain fundamental questions in physics for which there are no clear-cut answers. For example, what is time? I breathe. I can count the number of times I do that. Can we call those intervals as time? Suppose I stop breathing, does time seize to exist?*

Thank you!