Geometries and Constructions of Dynamical Maps including Dissipation

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Using finite closed algebra of operators for N-level systems to solve for time evolution

Sudarshan- Matthews- J. Rau, Phys. Rev. **121**, 920 (1961)

Time-dependent operator equations unitary integration

$$i\dot{U}(t) = H(t)U(t),\ U(0) = \mathcal{I}$$

$$\Psi(t) = U(t)\Psi(0) \qquad \rho(t) = U\rho(0)U^\dagger$$

$$U(t) \neq \exp[-i\int_0^t H(t')dt']$$

Time ordering
Dyson series

Unitary Integration $U(t) = \prod_j \exp[-i\mu_j(t)A_j],$

Baker-Campbell-Hausdorff Identity -- commutators of A

Phys. Lett. A **222**, 304 (1996) -- with K. Unnikrishnan
Phys. Rev. Lett. **81**, 4785 ('98) **79**, 5189 ('97) B A Shadwick and W F Buell J. Wei and E. Norman, J. Math. Phys. **4**, 575 (1963)

Illustration for a N=2j+1-level system in t-dependent fields

Example of a spin in a magnetic field $H(t) = -\vec{J}.\vec{B}(t)$

Closed algebra of three angular momentum operators J

Solution $$U(t) = e^{-i\mu_+(t)J_+}\, e^{-i\mu_- J_-}\, e^{-i\mu_3 J_3}$$

$$i\dot{U} = e^{-i\ldots J_+}\, \dot{\mu}_- J_-\, e^{-i\ldots J_-}\, e^{-i\ldots J_3} + \ldots$$

To move operators and re-arrange with three exp on the right,

Baker-Campbell-Hausdorff identity:

$$e^A B = (B + [A,B] + \frac{1}{2}[A,[A,B]] + \ldots)e^A$$

Cartesian J_x, J_y, J_z → Euler equations of rotation, highly nonlinear

Spin-1/2, SU(2)

$$U(t) = e^{z(t)\sigma_+/2} e^{w^*(t)\sigma_-/2} e^{-i\mu(t)\sigma_z/2}$$

form $\dot{U}$ Baker-Campbell-Hausdorff $(\quad)U$

$$\dot{z} = -i(H_{11} - H_{22} - zH_{21})z - iH_{12},$$

$$\dot{w}^* = i(H_{11} - H_{22} - 2zH_{21})w^* - iH_{21},$$

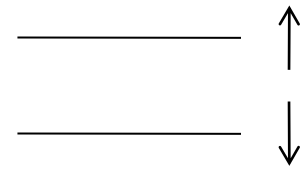
$$\dot{\mu} = H_{11} - H_{22} - 2zH_{21}.$$

Riccati, quadratically nonlinear equation for z by itself, other two solved by simple quadrature with z as input

Unitary by construction:

$$w^* = -z^*/(1 + |z|^2),\ e^{\mathrm{Im}\,\mu} = (1 + |z|^2)$$

Spin ½, Two-level system, Qubit



SU(2) Symmetry: 3 parameters

$$\begin{pmatrix} c_+ \\ c_- \end{pmatrix} \qquad |c_+|^2 + |c_-|^2 = 1$$

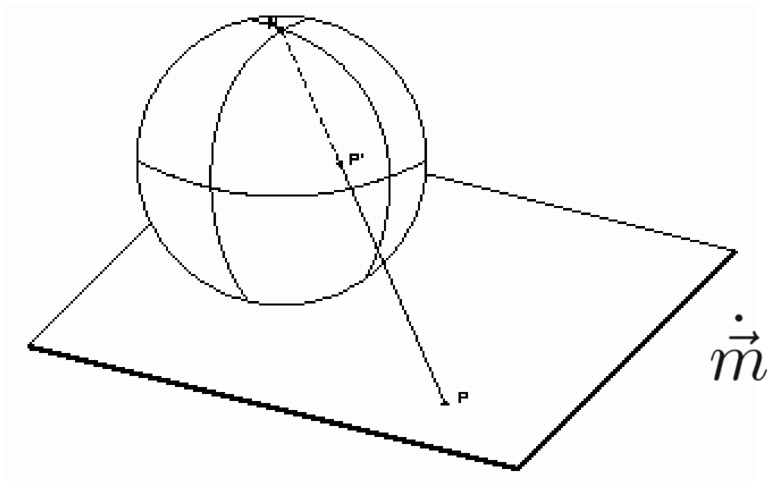
Evolution operator :

$$U(t) = e^{z(t)\sigma_+/2} e^{w^*(t)\sigma_-/2} e^{-i\mu(t)\sigma_z/2}$$

$$i\dot{\psi} = H(t)\psi(t), i\dot{\rho} = [H(t), \rho(t)]$$

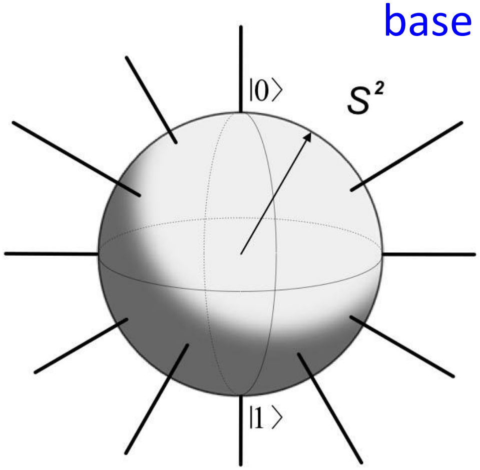
$$w^* = -z^*/(1+|z|^2), \; e^{\mathrm{Im}\,\mu} = (1+|z|^2)$$

Inverse stereographic projection of z to Bloch sphere S^2



$\mathrm{Re}\ \mu$: U(1) phase

$$\dot{\vec{m}} = -2\vec{B} \times \vec{m}$$



Dissipation and Decoherence

Master Equations -- GKLS Equation

Gorini-Kossakowski-Lindblad-Sudarshan

$$i\dot{\rho} = [H, \rho]$$

Liouville-von Neumann- Bloch

$$-\frac{1}{2}i\sum_k \left(L_k^\dagger L_k \rho + \rho L_k^\dagger L_k - 2L_k \rho L_k^\dagger\right)$$

Preserves trace, positivity of the density matrix

Complete Positivity

G. Lindblad, Commun. Math. Phys. **48**, 119 (1976)

V. Gorini, A. Kassakowski and E.C.G. Sudarshan, J. Math. Phys. **17**, 821 (1976) Stinespring 1955 Choi 1975 Kraus 1971

Chruscinski and Pascazio, arXiv:1710.05933 (2017)

U. Fano, RMP **29**, 74 ('57) Zwanzig, Wangsness-Bloch, Landau

$$\eta(t) = [\rho_{11} - \rho_{22}, \rho_{12} + \rho_{21}, i(\rho_{12} - \rho_{21})]$$

F.T.Hioe and J.H.Eberly, PRL **47**, 838 ('81), PRA **25**, 2168('82)

$$i\dot{\eta}(t) = \mathcal{L}(t)\eta(t)$$

Liouvillian superoperator

$\mathcal{L}(t)$ a 3 X 3 matrix

ARPR and R Wendell PRL **89**, 220405 (2002)

most general solution for $\eta(t)$ involves 8 exponentials

Full GKLS with arbitrary H(t) and L(t) solved, i.e.

a qubit with arbitrary dissipation and decoherence

N-level system: N^2 matrices/operators su(N) algebra

N^2 -1 μ_i equations

density matrix GKLS equation:(N^2 -1) X (N^2 -1) su(N^2 -1)

Symmetries, sub-groups, sub-algebras

Two qubit or 4-level su(4): su(3), so(5), su(2) X su(2) X u(1)

Construction for general SU(N)

Partition N-dim H as (N-n)- and n-dim blocks:

$$\mathbf{H}^{(N)} = \begin{pmatrix} \tilde{\mathbf{H}}^{(N-n)} & \mathbf{V} \\ \mathbf{V}^{\dagger} & \tilde{\mathbf{H}}^{(n)} \end{pmatrix}.$$

Write U again as three factors, first two nilpotent structure

$$\mathbf{U}^{(N)}(t) = \tilde{U}_1 \tilde{U}_2, \ \tilde{U}_1 = e^{\mathbf{z}(t)A_+} e^{\mathbf{w}^{\dagger}(t)A_-},$$

$$\tilde{U}_2 = \begin{pmatrix} \tilde{\mathbf{U}}^{(N-n)}(t) & \mathbf{0} \\ \mathbf{0}^{\dagger} & \tilde{\mathbf{U}}^{(n)}(t) \end{pmatrix}$$

$$i\dot{\tilde{U}}_2 = H_{\rm eff} \tilde{U}_2, \ H_{\rm eff} = \tilde{U}_1^{-1} H \tilde{U}_1 - i \tilde{U}_1^{-1} \dot{\tilde{U}}_1$$

$$i\dot{\mathbf{z}} = \tilde{\mathbf{H}}^{(N-n)} \mathbf{z} + \mathbf{V} - \mathbf{z}(\mathbf{V}^{\dagger} \mathbf{z} + \tilde{\mathbf{H}}^{(n)})$$

Matrix Riccati equation, **z**(0)=0. For n=1, **z**: (N-1) vector

Hamiltonian for (N-1) residual problem:

$$\mathbf{H}^{(N-1)} = \tilde{\mathbf{H}}^{(N-1)} - \frac{\mathbf{z}\mathbf{V}^\dagger + \mathbf{V}\mathbf{z}^\dagger}{\sqrt{\gamma}+1} - \frac{\mathbf{z}(\mathbf{z}^\dagger\mathbf{V} + \mathbf{V}^\dagger\mathbf{z})\mathbf{z}^\dagger}{2(\sqrt{\gamma}+1)^2}$$

explicitly Hermitian.

D. B. Uskov and ARPR: Phys. Rev. A **78**, 022331 (2008)

four-level/two-qubit SU(4): (z_1,z_2,z_3) Riccati, next (z_1,z_2), final z, three su-phases in all and six complex z's

vs

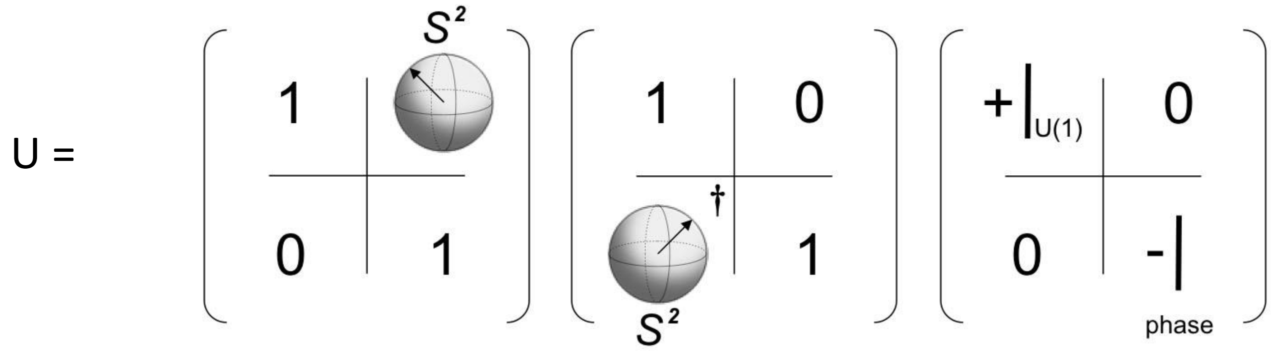
two-level/qubit SU(2): one phase and one complex z or S^2

SU(N): [SU(N)/(SU(N-1) X U(1))] X ((SU(N-1) X U(1))

base manifold fiber

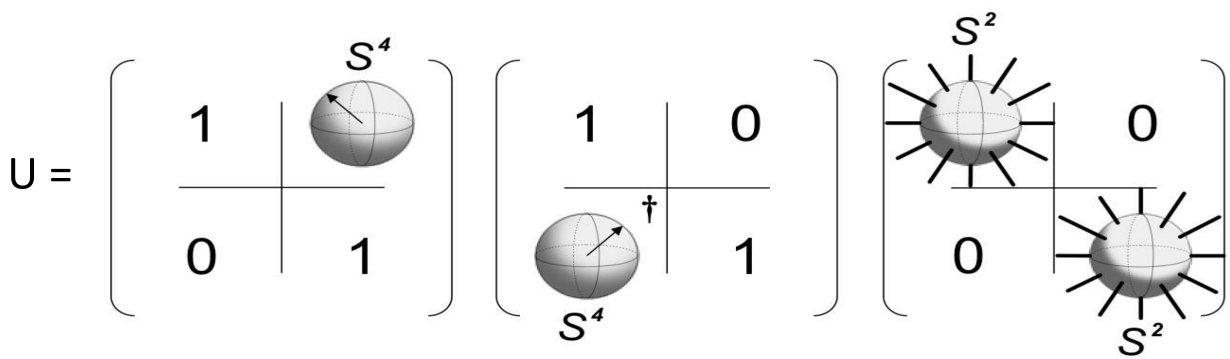
[SU(N)/(U(1) X U(1) .... U(1))] X (U(1) X U(1) ... U(1))

Schwinger philosophy (N-1) su-phases



$$SU(2) : S^2 \times U(1)$$

Bloch sphere base space X fiber, a single phase



$$H = \begin{pmatrix} (N-n)\times(N-n) & (N-n)\times n \\ n\times(N-n) & n\times n \end{pmatrix}$$

$$\downarrow$$

$$H_{eff} = \begin{pmatrix} (N-n)\times(N-n) & 0 \\ 0 & n\times n \end{pmatrix}$$

SU(4):Four-sphere base X fiber of two SU(2) manifolds

Sub-algebras of su(4)

su(3)   8 parameters $z : (z_1, z_2)$ plus one su(2) and u(1)

z a four-dimensional manifold

su(2) X su(2)   6 parameters   two su(2) fiber bundles, each an S^2 X U(1)

su(2) X su(2) X u(1)   7 parameters   two su(2) plus one u(1)

for each of the 15 operators, one such sub-algebra with that as the commuting element

so(5)   10 parameters   4-sphere plus two su(2) bundles

Full su(4): 15 parameters $z : (z_1, z_2, z_3, z_4)$ plus two su(2) and one u(1) z an eight-dimensional manifold

two spins qubit pair

I σ_i τ_i σ_i τ_j 15 operators su(4)

Table of Commutators

Each row has 7 zeroes, each commutes with 6 and anti- with 8

1

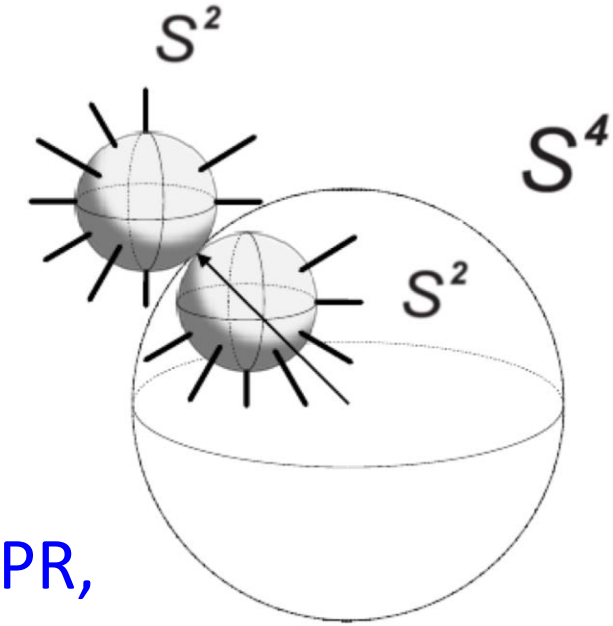
O_X	O_2	O_3	O_4	O_5	O_6	O_7	O_8	O_9	O_{10}	O_{11}	O_{12}	O_{13}	O_{14}	O_{15}	O_{16}
O_2	0	0	0	iO_6	$-iO_5$	iO_8	$-iO_7$	0	0	0	0	iO_{16}	$-iO_{15}$	iO_{14}	$-iO_{13}$
O_3	0	0	0	0	0	0	0	iO_{10}	$-iO_9$	iO_{12}	$-iO_{11}$	iO_{15}	$-iO_{16}$	$-iO_{13}$	iO_{14}
O_4	0	0	0	iO_8	$-iO_7$	$\frac{i}{4}O_6$	$-\frac{i}{4}O_5$	iO_{12}	$-iO_{11}$	$\frac{i}{4}O_{10}$	$-\frac{i}{4}O_9$	0	0	0	0
O_5	$-iO_6$	0	$-iO_8$	0	iO_2	0	iO_4	0	0	$-iO_{16}$	$-iO_{14}$	0	iO_{12}	0	iO_{11}
O_6	iO_5	0	iO_7	$-iO_2$	0	$-iO_4$	0	0	0	iO_{13}	iO_{15}	$-iO_{11}$	0	$-iO_{12}$	0
O_7	$-iO_8$	0	$-\frac{i}{4}O_6$	0	iO_4	0	$\frac{i}{4}O_2$	iO_{15}	$-iO_{13}$	0	0	$\frac{i}{4}O_{10}$	0	$-\frac{i}{4}O_9$	0
O_8	iO_7	0	$\frac{i}{4}O_5$	$-iO_4$	0	$-\frac{i}{4}O_2$	0	iO_{14}	$-iO_{16}$	0	0	0	$-\frac{i}{4}O_9$	0	$\frac{i}{4}O_{10}$
O_9	0	$-iO_{10}$	$-iO_{12}$	0	0	$-iO_{15}$	$-iO_{14}$	0	iO_3	0	iO_4	0	iO_8	iO_7	0
O_{10}	0	iO_9	iO_{11}	0	0	iO_{13}	iO_{16}	$-iO_3$	0	$-iO_4$	0	$-iO_7$	0	0	$-iO_8$
O_{11}	0	$-iO_{12}$	$-\frac{i}{4}O_{10}$	iO_{16}	$-iO_{13}$	0	0	0	iO_4	0	$\frac{i}{4}O_3$	$\frac{i}{4}O_6$	0	0	$-\frac{i}{4}O_5$
O_{12}	0	iO_{11}	$\frac{i}{4}O_9$	iO_{14}	$-iO_{15}$	0	0	$-iO_4$	0	$-\frac{i}{4}O_3$	0	0	$-\frac{i}{4}O_5$	$\frac{i}{4}O_6$	0
O_{13}	$-iO_{16}$	$-iO_{15}$	0	0	iO_{11}	$-\frac{i}{4}O_{10}$	0	0	iO_7	$-\frac{i}{4}O_6$	0	0	0	$\frac{i}{4}O_3$	$\frac{i}{4}O_2$
O_{14}	iO_{15}	iO_{16}	0	$-iO_{12}$	0	0	$\frac{i}{4}O_9$	$-iO_8$	0	0	$\frac{i}{4}O_5$	0	0	$-\frac{i}{4}O_2$	$-\frac{i}{4}O_3$
O_{15}	$-iO_{14}$	iO_{13}	0	0	iO_{12}	$\frac{i}{4}O_9$	0	$-iO_7$	0	0	$-\frac{i}{4}O_6$	$-\frac{i}{4}O_3$	$\frac{i}{4}O_2$	0	0
O_{16}	iO_{13}	$-iO_{14}$	0	$-iO_{11}$	0	0	$-\frac{i}{4}O_{10}$	0	iO_8	$\frac{i}{4}O_5$	0	$-\frac{i}{4}O_2$	$\frac{i}{4}O_3$	0	0

TABLE I: Table of commutators. With operators O_i in the first column and O_j in the top row, each entry provides the commutator $[O_i, O_j]$.

4-level systems, Two spins/qubits: SU(4) Symmetry,15 paramet

Sub-algebra of so(5) with 10 parameters

D.B.Uskov and ARPR,
Phys. Rev. A **74**,
030304 (R) (2006)
and **78**, 022331 (2008)



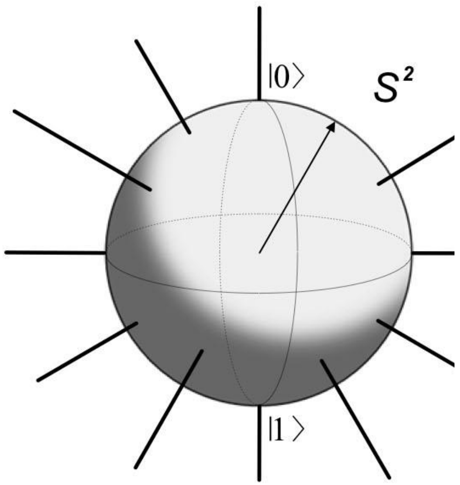
Sai Vinjanampathy and ARPR,
J. Phys. A **42**, 425303 (2009)

A four-sphere, the analog of Bloch two-sphere: 4 parameter
At each point on it, not a single phase as for single spin but t
"spiked Bloch spheres" for a total of six parameters.

Single qubit

SU(2)

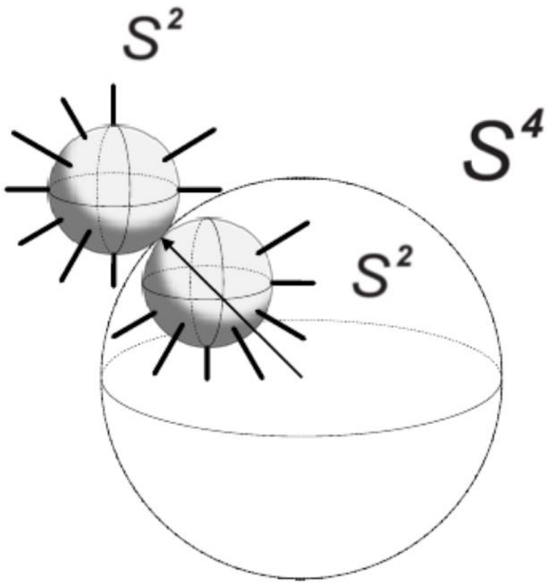
3 parameters   2 + 1



Two qubits

SO(5)

10 parameters   4 + 6

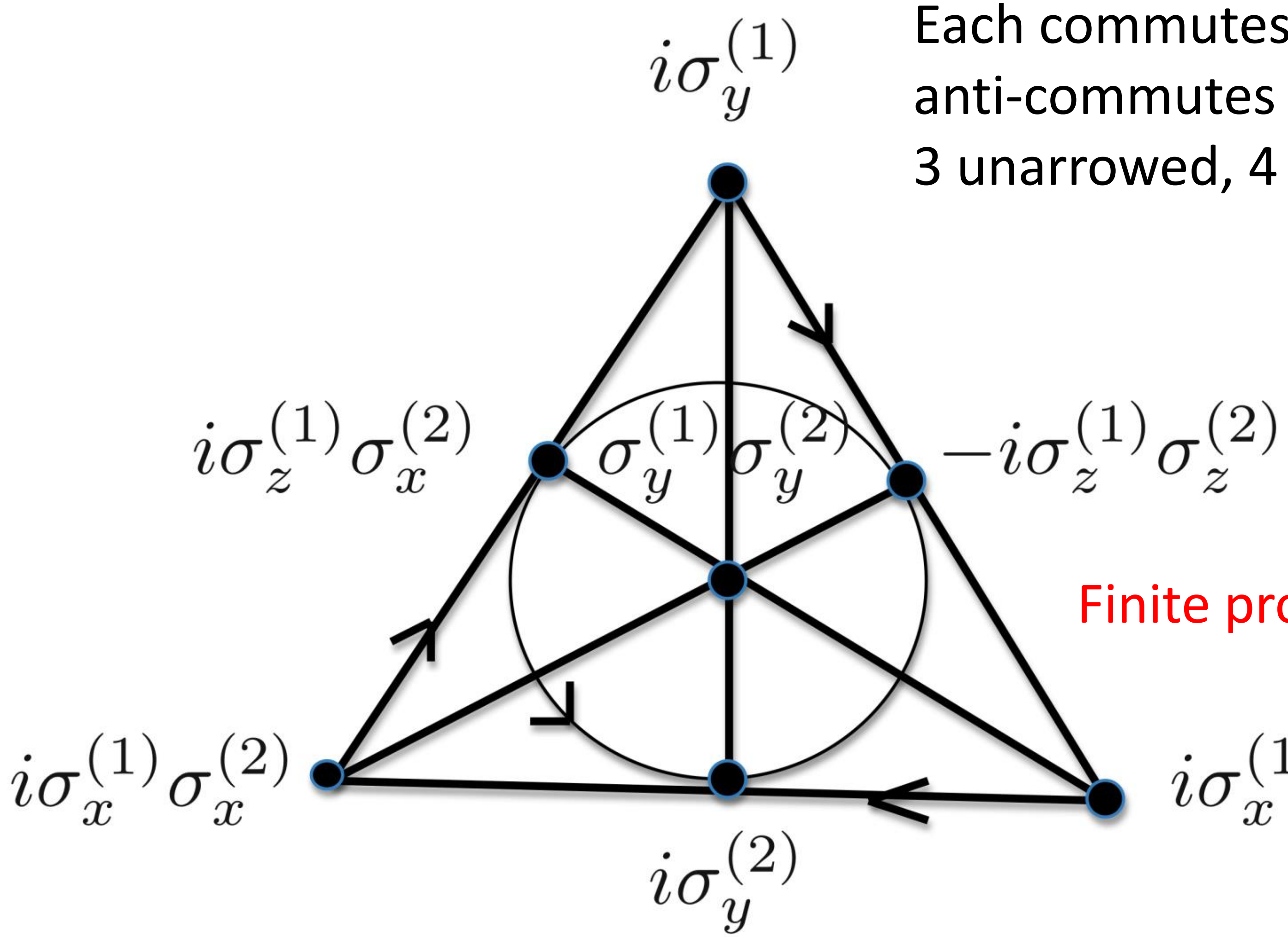


Direct generalization of base and fiber to higher dimensions

Subgroup of SU(4) : SU(2) X U(1) X SU(2)

Seven generators; choose any one of 15, 6 others : 7 in all

Each commutes with two others, anti-commutes with four
3 unarrowed, 4 arrowed lines



(Gino)Fano
Plane
PG(2,2)

Finite projective geometry

General two-qubit state has 15 parameters,
7 real along diagonal and 4 complex off-diagonal.
But, a restricted class called X-states with 7 para-meters, 3 real diagonal and 2 complex anti-diagonal, encompass pure/mixed, entangled/separable, etc.

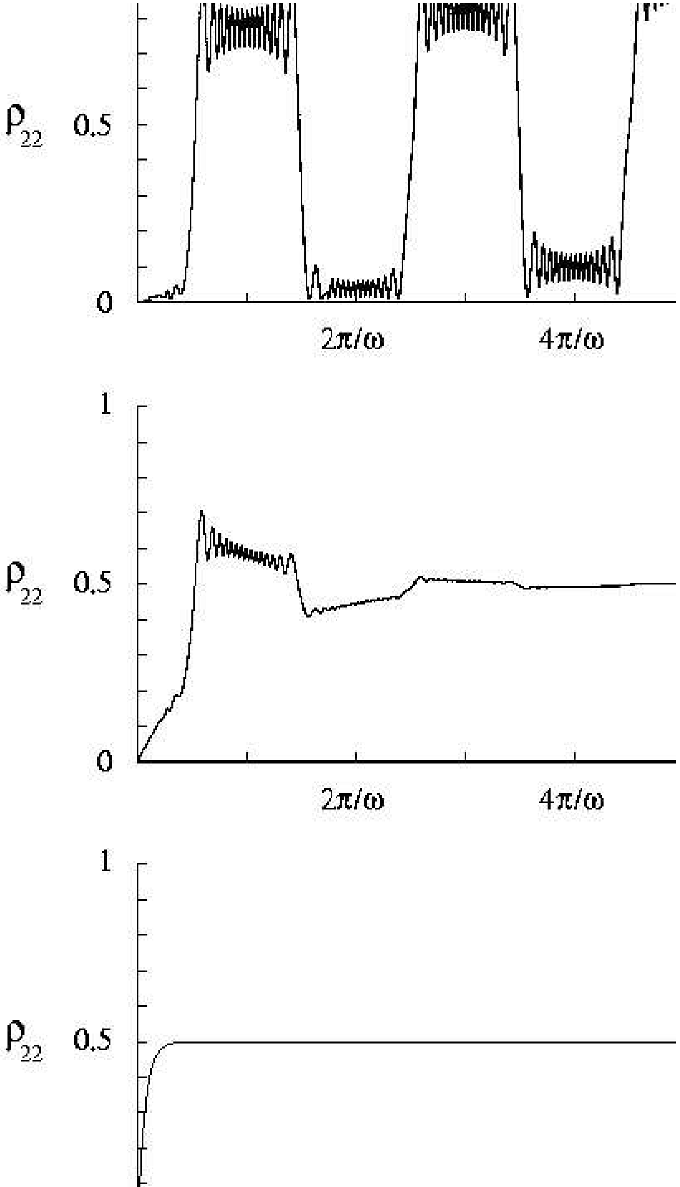
$$\begin{pmatrix} c_1 & 0 & 0 & z_1 \\ 0 & c_2 & z_2 & 0 \\ 0 & z_3 & c_3 & 0 \\ z_4 & 0 & 0 & c_4 \end{pmatrix}$$

Essence lies in su(2) X u(1) X su(2) symmetry
Fano Plane triangle of operators & states

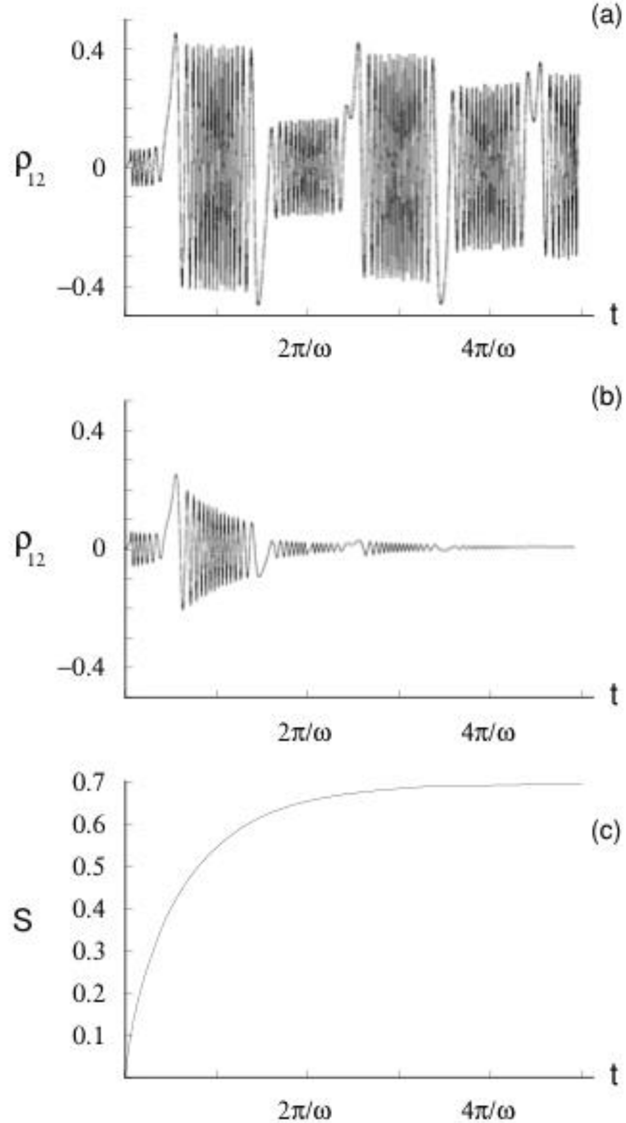
J P Marceaux and ARPR Quant. Inf. Proc. **19**, 49 ('19)
ARPR Symmetry **13**, 1732 (2021)

N = 2 with initial population in level 1

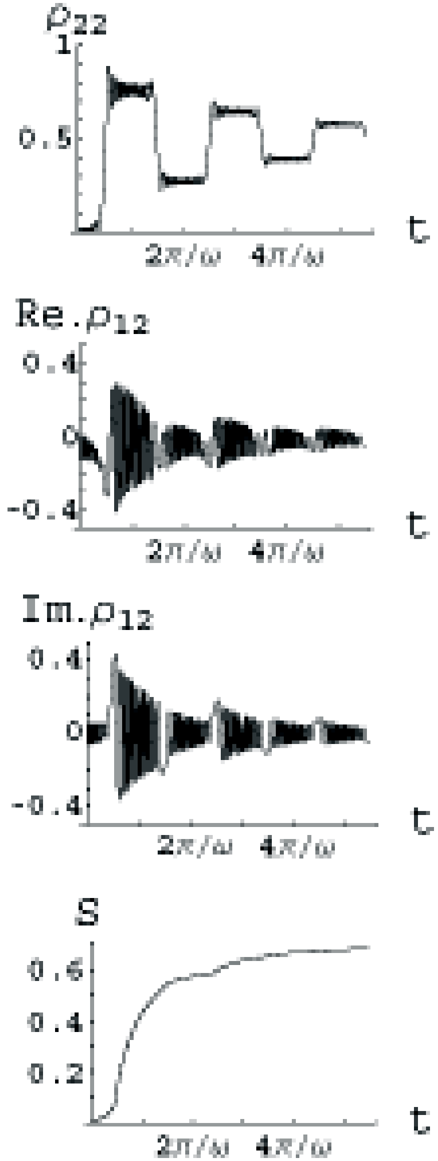
ARPR and R Wendell, PRL **89**, 220405 ('02)



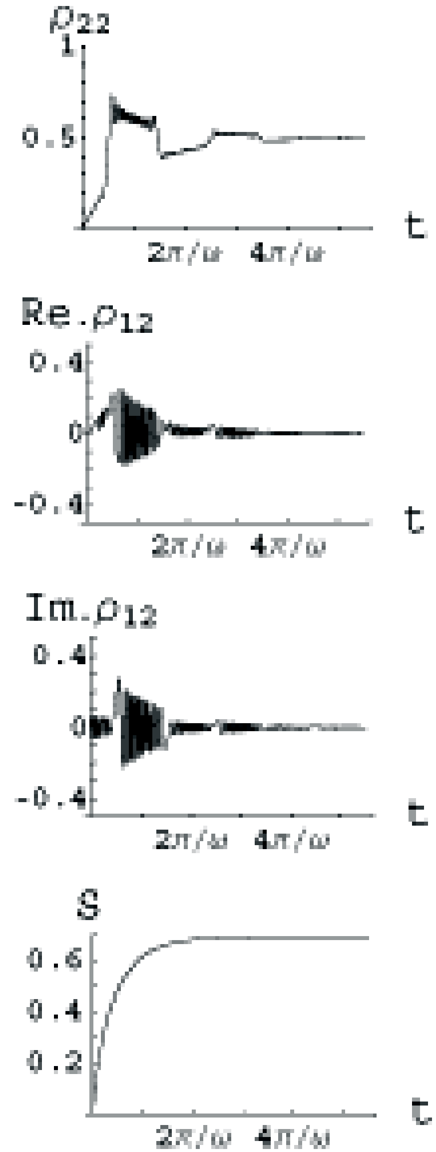
coherences



Entropy



Full 8 X 8 solution
Weichang Zhao



Simplified 3 X 3 model
Roger Wendell

Steady non-Hermitian decay plus quantum jump

initial excited state $|\psi\rangle$

wavefunction if photon emitted $a|\psi\rangle/\sqrt{\langle\psi|a^\dagger a|\psi\rangle}$

if no detection, evolve with $H = H_0 - (i\gamma/2)a^\dagger a$

$$i\dot{\psi} = H\psi, \psi(t+\delta t) = (1 - iH\delta t)\psi(t)$$

$$\langle\psi(t+\delta t)|\psi(t+\delta t)\rangle = 1 - \gamma\delta t\langle\psi|a^\dagger a|\psi\rangle = 1 - \delta\rho$$

normalizing non-emission, $\psi(t+\delta t) = (1 - iH\delta t)\psi/\sqrt{1-\delta\rho}$

$$\rho = |\psi(t+\delta t)\rangle\langle\psi(t+\delta t)| = P_{emi}|\psi_{emi}\rangle\langle\psi_{emi}| + P_{non}|\psi_{non}\rangle\langle\psi_{non}|$$

$$= \gamma\delta t a\rho a^\dagger + \rho(t) - i\delta t H\rho + i\delta t\rho H^\dagger$$

$$\frac{d\rho}{dt} = \frac{\rho(t+\delta t) - \rho(t)}{\delta t} = -i[H_0, \rho] + L\rho$$

$$L = \gamma a\rho a^\dagger - (\gamma/2)a^\dagger a\rho - (\gamma/2)\rho a^\dagger a$$