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What are the possible near field structures one can define over the multiplicative group of a near field?

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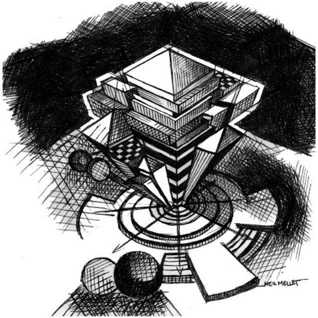
Nithecs colloquium, 13 February 2023



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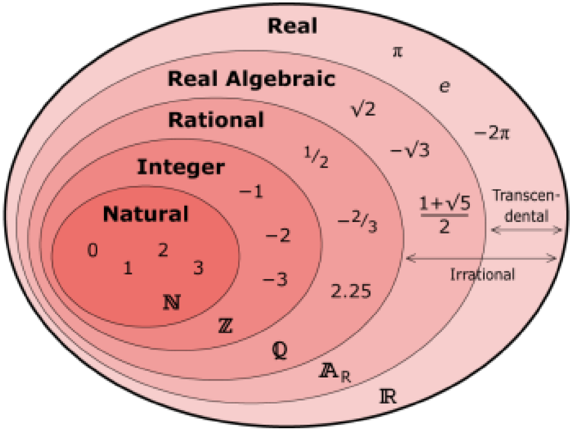
The field of real numbers.



How to understand the addition of natural numbers.

1. Everything starts with the notion of ONE (eg. 3 billion years ago).
2. Abstractly, define the concept of 2 to be the same as the abstract interpretation that one has of the sum of $1 + 1$. (Those are really just symbols we put on a abstract concept we are trying to formalize.)
3. Similarly, you define the concept of 3 to be the same as the abstract interpretation that one has of the sum $1 + 1 + 1$. But now you also conceptually know that $2 + 1$ and $1 + 2$ should also be 3, from your abstract understanding of those numbers. That translates into mathematics as what we refer to as the associativity property of natural numbers which is

$$1 + (1 + 1) = (1 + 1) + 1$$

4. Recursively, conceptually, you define the sequence of natural numbers and their corresponding equivalent addition interpretations.

Reflecting on the definition of the addition on the natural numbers.

Natural number were discovered by Babylonians, somewhere between 2000 BC and 1500 BC.

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The zero symbol.

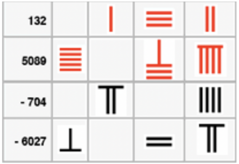
A zero-like symbol dates to sometime around the third century B.C. in ancient Babylon.

This symbol was revolutionary, especially for the binary computer system, but also for writing numbers effectively.

In mathematics, 0 has a special place that is $0 + n = n$ for any n natural number. The zero has no effect on the addition.

Integers?

In 200 BCE the Chinese number rod system represented positive numbers in Red and Negative numbers in black. These were used for commercial and tax calculations where the black canceled out the red.



How can we add integers?

1. We can understand the concept of -1 as the concept that ”validates” conceptually the string of symbols $-1 + 1 = 0 = 1 - 1$.
2. We do the process as above and define -2 to be the symbol $-1 - 1$.
3. You make sense of any negative number as we did before for natural numbers.
4. To add a negative number to a positive number you can think of them as the sum of 1 and -1. You then collect the pairs of 1 and -1 together, those are zero and you know how it affect a sum. At the end of this process, you have either zero, or a string of positive numbers, or a string of negative numbers then you know what to do!

Reflection on integers.

This addition is commutative: $n + m = m + n$ **for all** n **and** m **natural number.**
Negative numbers make equations of the form $x + n = 0$ **where** n **is a natural number have solutions.**

Rational numbers?

Rational numbers were invented in the sixth century BCE. Rational number, could be seen as a bit more "continuous" than numbers.
1. In the sense, you would need to think about a unit (sometimes length) that you also name "1". **So now the concept of discrete concept of ONE is evolving and growing into a more continuous version of the same concept.**
2. You cut in q parts of this 1 as you now understand what a natural number. You want to consider p portions of these cut-off sections. You understand this concept as p/q. If p is bigger than q you might need more than one length to have p of them so that $1 + p/q$ is the same as $(p+q)/q$ for instance. If you have negative numbers you need to combine the understanding of integers with this understanding of things after grouping the sign on the numerator using the sign rules $(+) \cdot (+) = (+)$, $(+) \cdot (-) = (-)$, $(-) \cdot (+) = (-)$ and $(-) \cdot (-) = (+)$.

Rational numbers?

Rational numbers are starting to fill up the real line.
Natural numbers are rational numbers once you decide that n and $n/1$ are conceptually the same.
Rational numbers make equations of the form $px - q = 0$ where p and q are integers have solutions.

How to add the rational numbers?

Now if you are adding p/q to r/s
1. you start combining the sign into the numerator.
2. you need first to have a common cut-off length to be able and go ahead to understand how to add based on how we understood addition from the integers. Because it does not easy to add non-equal portions right?
3. We can then cut our unit in qs portions,
but now p/q would be ps of this new cut-off portion that is ps/qs
and r/s will be qr of this new cut-off portion that is rq/qs.
So that we have $rs + ps$ cut-off-portion by qs
4. that is what we understand the addition $p/q + r/s$ as the symbol $(rs + ps)/qs$.

Reflecting on the addition of irrational numbers.

Hippassus of Metapontum, a Greek philosopher of the Pythagorean school of thought, is widely regarded as the first person to recognize the existence of irrational numbers.

Irrational number?

It was proven that $\sqrt{2}$ which is the symbol understood as the place holder x that solves the equation $x^2 - 2 = 0$ was irrational (not rational).

How to add irrational numbers?

It can be proven using the archimedean property of $\mathbb{R}$ that any irrational number is what we understand as the limit of a sequence of rational numbers p_n/q_n;
We understand a sum of two real numbers as the limit of the sum of the corresponding sequences of rational numbers.

Complex numbers?

French mathematician René Descartes was the first to emphasize the imaginary nature of numbers.
However, the conceptualization of complex numbers dates back to the 16th century with the contribution of Italian mathematician Gerolamo Cardano.
Later, in the 18th century, mathematician Carl Friedrich Gauss consolidated Cardano's premises.

How to add complex numbers?

1. Reflecting on the equation $x^2 + 1 = 0$ having no solution.
2. We can imagine i to be a solution.
3. We define a number $a + ib$ where a and b are real numbers, that one can conceptually understand to be a new form of number.
4. We decide according to our understanding that such a ”formal sum” $a + ib$ is zero when $a = 0$ and $b = 0$.
5. We can define $(a + ib) + (c + id) = (a + c) + i(b + c)$.

How to define the multiplication of two integers?

1. We define the multiplication $n \cdot m$ where n and m are natural number to be the repeated addition $m + ... + m$, n times but this happens to be also $n + ... + n$, m-times. We observe that we can rewrite this as $n(1 + ... + 1) = n + .. + n = (1 + ... + 1)n$. This is the distributivity of the multiplication onto the addition.
2. We set $(-1)x = sign(x)x = -x$
3. We define the multiplication $n \cdot m$ where n and m are integer by determining the sign of $n \times m$ following the sign rule and then setting $n \cdot m = sign(mn)|n| \cdot |m|$ where $|n|$ is understood as n if n was already a positive numbers and $-n$ otherwise.

How to define the multiplication of two (real) complex numbers?

3. We define the multiplication $p/q \cdot r/s$ where p/q and r/s are rational numbers to be the repeated addition $p/qs + ... + p/qs$, r times but this happens to be also $r/qs + ... + r/qs$, p times.
4. We define the multiplication $x \cdot y$ where x and y are real number numbers as the limit of the multiplication of the chosen rational number sequences associated with x and y.
5. We define the multiplication $(a + ib)(c + id)$ as $a(c + id) + ib(c + id)$ but also $ac + iad + ibc - bd = (ac - bd) + i(ad + bc)$, using the distributivity laws of the multiplication on the addition that were already observed when watching closely the property of the multiplication on real numbers.

Take home

The conceptual understanding of the addition of natural numbers induces the addition and multiplication of real numbers and complex numbers in a continuous way.

There seems to be always an equation behind a number system. Mathematics is not built in one day.

Conceptual understanding even of a single concept evolve as our knowledge grows. Never underestimate one concept.

Be open-minded.

Why $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are rings?

A unitary ring is a set R equipped with two binary operations $+$ (addition) and $\cdot$ (multiplication) satisfying the following three sets of axioms, called the ring axioms

1 - R is an abelian group under addition, meaning that:

- $(a+b)+c = a+(b+c)$ for all a, b, c in R (that is, $+$ is associative)

- $a+b = b+a$ for all a, b in R (that is, $+$ is commutative).

- There is an element 0 in R such that $a+0=a$, for all a in R (that is, 0 is the additive identity).

- For each a in R there exists $-a$ in R such that $a+(-a)=0$ (that is, $-a$ is the additive inverse of a).

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Why $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are unitary rings?

2 - R is a monoid under multiplication, meaning that:
- $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all a, b, c in R (that is, $\cdot$ is associative).
- There is an element 1 in R such that $a \cdot 1 = a$ and $1 \cdot a = a$ for all a in R (that is, 1 is the multiplicative identity).
3- Multiplication is distributive with respect to addition, meaning that:
- $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all a, b, c in R (left distributivity).
- $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all a, b, c in R (right distributivity).

Why $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are fields?

A field is a set F together with two binary operations on F called addition and multiplication such that $(F, +, \cdot)$ is a ring and every element has a multiplicative inverse: for every a in F non-zero, there exists an element in F, denoted by a^{-1} or $1/a$, called the multiplicative inverse of a, such that $a \cdot a^{-1} = 1$.

How could we interpret the multiplication without the addition in $\mathbb{Q}$?

To understand the multiplication of $\mathbb{Q}$ without the addition, we can look at the prime numbers and the units. Indeed, $\mathbb{Z}$ is a unique factorization domain. That is, every integer can be written uniquely as a product of primes up to units.

Question: Is it possible to define a meaningful addition compatible with the multiplication?

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UNDERLYING QUESTIONS

Let $(F, +, \cdot)$ be a right near-field. We forget the additive structure of the field.

- What are the additive structures $\boxplus$ such that $(F, \boxplus, \cdot)$ is a field?
- What can we say about $(F, \boxplus, \cdot)$?
- What can we learn from these additions?

In this talk, for simplicity purposes, we will focus on $\mathbb{R}$ and $\mathbb{C}$. Some of the content will apply to general near-fields even near rings.

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Multiplicative additive structure (LB, SM)

Let $(K, +, \cdot)$ be a right near-field and ϕ be an automorphism of K^* as a multiplicative group. We define the right near-field induced by ϕ, denoted by $(K, +_\phi, \cdot)$ or simply K_ϕ, as the right near-field with the same multiplication as $(K, +, \cdot)$ and an induced addition given by

$$a +_\phi b = \phi^{-1}(\phi(a) + \phi(b))$$

for any $a, b \in K$,where ϕ is extended to K by setting $\phi(0) = 0$.

ϕ is a right near-field isomorphism between K_ϕ and K.
Given ϕ and ψ, two multiplicative automorphisms, we have $K_\phi = K_\psi$ if and only if $\psi \circ \phi^{-1}$ is a right near-field automorphism of $(K, +, \cdot)$. When $K = \mathbb{R}$ or $\mathbb{C}$ and ϕ is continuous, then ϕ is a continuous isomorphism. In that case, we refer to K_ϕ as a continuous field associated to ϕ.

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Real numbers

Theorem

The continuous automorphisms of $(\mathbb{R}^*, \cdot)$ are precisely given by

$$\begin{array}{rccc} \epsilon_\alpha : & \mathbb{R}^* & \to & \mathbb{R}^* \\ & x & \mapsto & \left\{ \begin{array}{cc} x^\alpha & x > 0 \\ -(-x)^\alpha & x < 0 \end{array} \right. \end{array}$$

for $\alpha \in \mathbb{R}^*$. Moreover, the inverse of ϵ_α, ϵ_α^{-1} equals $\epsilon_{1/\alpha}$ and is therefore ϵ_α is a homeomorphism.

Complex numbers

Theorem (LB, B. Blum Smith, SM)

The continuous automorphisms of $(\mathbb{C}^*, \cdot)$ are either of the form

$$\begin{array}{rccc} \epsilon_\alpha : & \mathbb{C}^* & \to & \mathbb{C}^* \\ & z = rs & \mapsto & r^\alpha s \end{array} \quad \text{or} \quad \begin{array}{rccc} \overline{\epsilon_\alpha} : & \mathbb{C}^* & \to & \mathbb{C}^* \\ & z = rs & \mapsto & r^\alpha \overline{s} \end{array}$$

where $r \in \mathbb{R}_{>0}$, $s \in \mathbb{S}$ and $\alpha \in \mathbb{C} \backslash i\mathbb{R}$.

$\overline{\epsilon_1}$ is the complex multiplication, $\overline{\epsilon_\alpha} = \epsilon_\alpha \circ \overline{\epsilon_1}$, $\epsilon_\alpha = \overline{\epsilon_\alpha} \circ \overline{\epsilon_1}$, $\overline{\epsilon_1} \circ \epsilon_\alpha = \overline{\epsilon_{\overline{\alpha}}}$ and $\overline{\epsilon_1} \circ \overline{\epsilon_\alpha} = \epsilon_{\overline{\alpha}}$. Finally, the inverse of ϵ_α (resp. $\overline{\epsilon_\alpha}$), ϵ_α^{-1} (resp. $\overline{\epsilon_\alpha}^{-1}$) is $\epsilon_{\frac{1 - i\,\mathrm{Im}(\alpha)}{\mathrm{Re}(\alpha)}}$ (resp. $\overline{\epsilon_{\frac{1 + i\,\mathrm{Im}(\alpha)}{\mathrm{Re}(\alpha)}}}$) and therefore ϵ_α is a homeomorphism.

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Example 1

We denote the addition $m +_{\epsilon_\alpha} n$ simply as $m +_\alpha n$.
For $K = \mathbb{R}$ we take $m = 1$, $n = 2$ and consider the sum $m +_\alpha n$, where $\alpha \in \mathbb{R}$.
For $\alpha = 1$, we know the answer.
For $\alpha = 2$, we have $\epsilon_2(x) = x^2$ for $x > 0$ and $\epsilon_2(x) = -(-x)^2$ for $x < 0$. Then

$$\begin{aligned} 1 +_2 2 &= \epsilon_2^{-1}\left(\epsilon_2(1) + \epsilon_2(2)\right) \\ &= \sqrt{1^2 + 2^2} \\ &= \sqrt{(1+4)} \\ &= \sqrt{5} \end{aligned}$$

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Example 1 continued

For $\alpha = 3$, we have $\epsilon_3(x) = x^3$ and

$$\begin{aligned} 1 +_3 2 &= \epsilon_3^{-1}\left(\epsilon_3(1) + \epsilon_3(2)\right) \\ &= \left(1^3 + 2^3\right)^{1/3} \\ &= (1+8)^{1/3} \\ &= 9^{1/3} \end{aligned}$$

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Example 1 continued

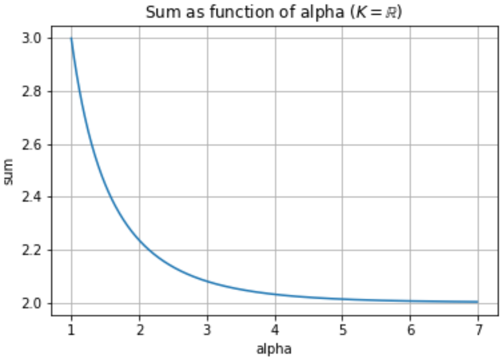
One more example. Suppose, $m = 1$, $n = -2$, then for, say $\alpha = 2$, we would get

$$\begin{aligned} 1 +_2 (-2) &= \epsilon_2^{-1}\left(\epsilon_2(1) + \epsilon_2(-2)\right) \\ &= \epsilon_2^{-1}(1 - 2^2) \\ &= \epsilon_2^{-1}(-3) \\ &= -\sqrt{3} \end{aligned}$$

Note that the result is NOT $\sqrt{-3}$ which would not make sense on $\mathbb{R}$. This follows from the definition of the automorphism.

Example 1 continued

The result is valid for $\alpha \in \mathbb{R}$, so plotted for different values of α we have



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Example 2

Reminder: $\epsilon_\alpha(z) = |z|^\alpha e^{i\arg(z)}$ and $\epsilon_\alpha^{-1}(z) = \epsilon_{\frac{1-i\,\mathrm{Im}(\alpha)}{\mathrm{Re}(\alpha)}}(z) = |z|^{\frac{1-i\,\mathrm{Im}(\alpha)}{\mathrm{Re}(\alpha)}} e^{iarg(z)}$.

For $K = \mathbb{C}$, suppose $m = 1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$ and $n = 3 + 3i = \sqrt{18}e^{i\frac{\pi}{4}}$ and consider the sum $m +_\alpha n$, where $\alpha = a$, where $a \in \mathbb{R}$.

We have

$$\begin{aligned} m +_\alpha n &= \epsilon_\alpha^{-1}\left(\epsilon_\alpha(m) + \epsilon_\alpha(n)\right) \\ &= \epsilon_\alpha^{-1}\left((\sqrt{2})^a e^{i\frac{\pi}{4}} + (\sqrt{18})^a e^{i\frac{\pi}{4}}\right). \end{aligned}$$

For $\alpha = a$, we have $\mathrm{Im}(\alpha) = 0$. Therefore, $\epsilon_\alpha^{-1}(z) = |z|^{\frac{1}{a}} e^{iarg(z)}$.

For, say, $a = 2$, we have

$$\begin{aligned} m +_\alpha n = \epsilon_2^{-1}\left((\sqrt{2})^2 e^{i\frac{\pi}{4}} + (\sqrt{18})^2 e^{i\frac{\pi}{4}}\right) &= \epsilon_2^{-1}\left((2+18)e^{i\frac{\pi}{4}}\right) \\ &= (20)^{\frac{1}{2}} e^{i\frac{\pi}{4}} \\ &\approx 3,16 + 3,16i \end{aligned}$$

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Example 3

Reminder: $\epsilon_\alpha(z) = |z|^\alpha e^{i\arg(z)}$ and $\epsilon_\alpha^{-1}(z) = \epsilon_{\frac{1-i\,\mathrm{Im}(\alpha)}{\mathrm{Re}(\alpha)}}(z) = |z|^{\frac{1-i\,\mathrm{Im}(\alpha)}{\mathrm{Re}(\alpha)}} e^{iarg(z)}$.

Let us again consider $m = 1 + i = \sqrt{2}\, e^{i\frac{\pi}{4}}$, $n = 3 + 3i = \sqrt{18}\, e^{i\frac{\pi}{4}}$ and consider the sum $m +_\alpha n$, but now $\alpha = a + ib$, where $a, b \in \mathbb{R}$.

We have

$$\begin{aligned} m +_\alpha n &= \epsilon_\alpha^{-1}\left(\epsilon_\alpha(m) + \epsilon_\alpha(n)\right) && (1) \\ &= \epsilon_\alpha^{-1}\left((\sqrt{2})^{a+ib}\, e^{i\frac{\pi}{4}} + (\sqrt{18})^{a+ib}\, e^{i\frac{\pi}{4}}\right) && (2) \end{aligned}$$

Recall that $s^{a+ib} = s^a s^{ib} = s^a e^{ib\ln s}$. For, say, $a = 2, b = 1$, we will then have

$$\epsilon_2^{-1}\left((\sqrt{2})^2 e^{i(\frac{\pi}{4}+\ln\sqrt{2})} + (\sqrt{18})^2 e^{i(\frac{\pi}{4}+\ln\sqrt{18})}\right) \approx 3,43 + 2,69i \qquad (3)$$

(Obviously, we skipped a few steps of grinding calculation between the second last line and the final answer. It is tedious, so it is best done numerically.)

Comment on Examples 2 and 3

The geometric effect of the automorphism of the complex numbers can be seen by looking closer at elements of these two examples;

- For the case where $\alpha = a$, we saw $\epsilon_\alpha(1+i) = (\sqrt{2})^a e^{i\frac{\pi}{4}}$. In other words, only a "stretching" of the modulus.
- For the case where $\alpha = a + ib$, we saw $\epsilon_\alpha(1+i) = (\sqrt{2})^a e^{i(\frac{\pi}{4}+b\ln\sqrt{2})}$. In other words, a "stretching" of the modulus and a rotation.

Transformation of the Complex Plane

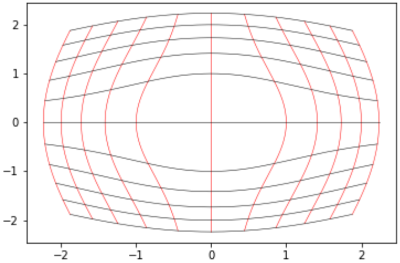

Figure: Transformation when $\alpha = 2$.

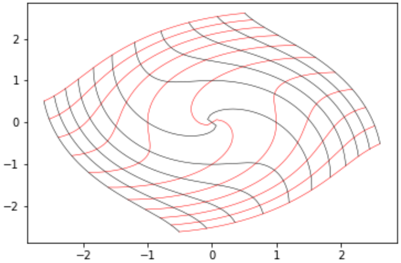

Figure: Transformation when $\alpha = 2 + i$.

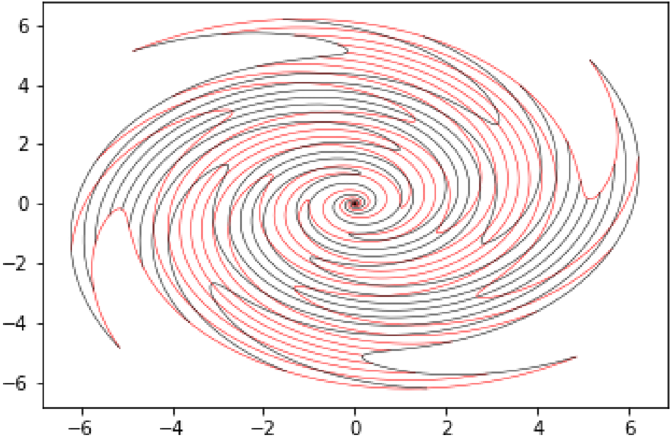

Figure: Transformation when $\alpha = 1 - 4i$

Transformation of the Complex Plane

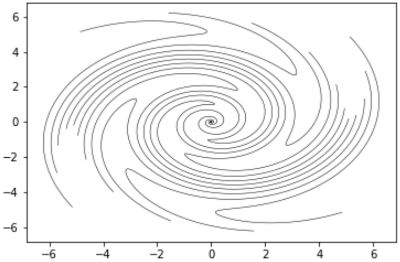

Figure: Transformation of horizontal lines when $\alpha = 1 - 4i$.

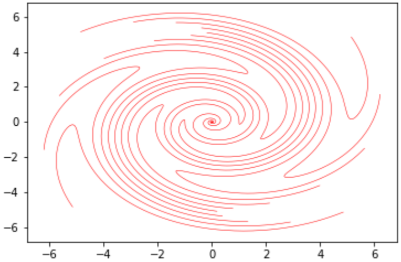

Figure: Transformation of vertical lines when $\alpha = 1 - 4i$.

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Classical power means

The additions induced by the multiplicative automorphisms allow us to define an extension of the power means to include datasets with negative real numbers, as well as the equivalent means for complex numbers.
Recall the usual definition of the weighted power mean of a set of n positive real numbers, say r_j with weight ω_j, for all $j \in \{1, \ldots, n\}$ such that $\sum_{j=1}^{n} \omega_j = 1$, is

$$M_\alpha((r_1, \omega_1), \ldots, (r_n, \omega_n)) = \left(\sum_{j=1}^{n} \omega_j r_j^\alpha \right)^{\frac{1}{\alpha}},$$

where $\alpha \in \mathbb{R}$.

Towards generalizing power means

Let us write the weights, ω_j as $\omega_k = e^{i\theta_k} + e^{-i\theta_k}$, where $\theta_k = \arccos(w_k/2)$. Defining $a_{j,k} = r_j e^{ik\theta_j}$, where $j \in \{1, \ldots, n\}$ and $k \in \{\pm 1\}$, we can show that the usual power mean for exponent $\alpha \in \mathbb{R}$ of the set of real numbers can be written as

$$\begin{aligned}
\sum_{j=1}^{n} (a_{j,1} +_\alpha a_{j,-1}) &= \sum_{j=1}^{n} \left(r_j e^{i\theta_j} +_\alpha r_j e^{-i\theta_j} \right) \\
&= \epsilon_\alpha^{-1} \left(\sum_j \left(r_j^\alpha e^{i\theta_j} + r_j^\alpha e^{-i\theta_j} \right) \right) \\
&= \epsilon_\alpha^{-1} \left(\sum_j r_j^\alpha (e^{i\theta_j} + e^{-i\theta_j}) \right) \\
&= \epsilon_\alpha^{-1} \left(\sum_j \omega_j r_j^\alpha \right) \\
&= \left(\sum_j \omega_j r_j^\alpha \right)^{\frac{1}{\alpha}} \\
&= M_\alpha((r_1, \omega_1), \ldots, (r_n, \omega_n))
\end{aligned}$$

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Generalized power means (LB, SM)

Using the same procedure, we can construct the power mean for datasets that consist of both positive and negative real numbers, purely negative numbers, or complex numbers.
Very briefly, assuming $\alpha \in \mathbb{R}$, one writes $r_j = |r_j| e^{iarg(r_j)}$ and follow the same procedure as above to obtain

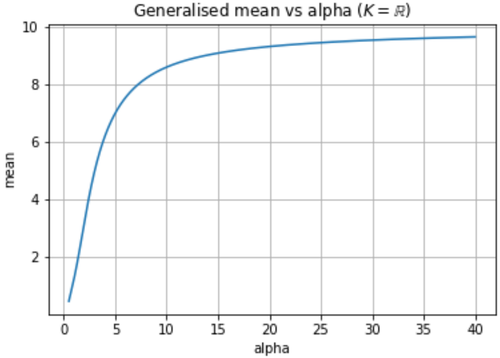
$$M_\alpha((r_1, \omega_1), \ldots, (r_n, \omega_n)) = \epsilon_\alpha^{-1} \left(\sum_j \omega_j |r_j|^\alpha e^{iarg(r_j)} \right).$$

This is a more general form of a power mean, which includes the possibility of computing one for complex-valued data points. The normal case of all $r_j > 0$ is obtained when all $arg(r_j) = 0$. Negative numbers in the dataset are those for which $arg(r_j) = \pi$.
In principle, one could extend this idea to complex-valued α and weights, but the (physical) interpretation of precisely what these represent is unclear.

Example

We compute the generalized power mean of the set $\{1, 2, -8, 10\}$. That is, the weighted power mean with respect to equal weights. We obtain

$$M_\alpha\left(\left(1, \frac{1}{4}\right), \left(2, \frac{1}{4}\right), \left(-8, \frac{1}{4}\right), \left(10, \frac{1}{4}\right)\right) = \epsilon_\alpha^{-1}\left(\frac{1}{4}\left(1^\alpha + 2^\alpha - (-8)^\alpha + 10^\alpha\right)\right).$$



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Something nice to end, we construct an addition $\boxplus$ on $\mathbb{Q}$ such that $(\mathbb{Q}, \boxplus, \cdot)$ is isomorphic to $(\mathbb{Q}(\sqrt{-19}), +, \cdot)$!

Thank you for listening!